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Progenitors Related to Simple Groups

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Progenitors Related to Simple Groups

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Abstract

This thesis contains methods of finding new presentations of finite groups, particularly nonabelian simple groups. We have presented several progenitors such as $2^{*10} : D_{10}$, $2^{*8} : \mathbb{Z}_4 \wr \mathbb{Z}_2$, $3^{*3} :_m L_2(3)$, $2^{*7} : L_2(7)$, $2^{*4} : [2 : 2^2]$, $2^{*11} : D_{11}$ and many more on which we've found the mathieu group M_{12}, and $2^{\bullet}[M_{21} : 2^2]$ among their homomorphic images. We give the full monomial automorphism groups of $Aut(3^{*2})$, $Aut(3^{*3})$, and $Aut(5^{*2})$. Included is a proof showing that the full monomial automorphism group of $Aut(m^{*n})$ is isomorphic to $U(m) \wr S_n$. In addition we have constructed the Cayley Diagrams of $PGL_2(7)$, $[3 \times A_5] : 2$, $3 : [A_6 : 2]$, and $2 \times [(3 \times L_2(11)) : 2]$ using the process of double coset enumeration.

Acknowledgements

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Table of Contents

List of Tables

List of Figures

Introduction

A fundamental objective of group theory is to find and classify all finite simple groups. The composition factors of a finite group allow us to view our group as a product of its simple subgroups. By solving the extension problem we are able to find a homomorphic image of the finite group. By completing the double coset enumeration of a finite group we are able to construct the finite homomorphic image of the group by finding all of its double cosets and forming its Cayley Diagram. To find finite simple groups we write presentations for progenitors and factor those progenitors by relations. There are various techniques for writing the presentation for a progenitor of which we discuss in detail in later chapters. A few methods we highlight in this thesis include wreath product progenitors, transitive progenitors, and monomial progenitors. Once we write the progenitosr we add relations to each and find their homomorphic images.

The process of finding these progenitors and homomorphic images was made possible with the aid of MAGMA Computational Algebra System, [**?**]. MAGMA is a large software package that performs computational algebra, number theory, algebraic geometry, and algebraic combinatorics. Its mathematically rigorous environment allows us to define and work with group structures necessary for our field of research.

We begin Chapter 1 providing key theorems and definitions that are used throughout this thesis. In Chapter 2 we first discuss the process of Double Coset Enumeration resulting in the construction of an isomorphic image of the Cayley Diagram. Chapter 3 tackles the extension problem in which we look at a group's composition factors and find the group's isomorphic type. In Chapter 4 we describe the process of creating Wreath Product progenitors, adding in relations and the resulting finite non-abelian simple groups. Next we explore Monomial Progenitors in Chapter 5 and prove that the full automorphism group of m^{*n} is the wreath product $U(m) \wr S_n$. In Chapter

6 we show how we construct Monomial progenitors. Similarly, in Chapter 7 we describe the process of creating Transitive progenitors. Finally, in Chapter 8 we list additional progenitors that we have found over the course of our research and their tables of simple groups.

Chapter 1

Preliminaries

Before we can begin to discuss the process of our work and our findings we must first discuss the preliminaries. In this chapter we will focus on definitions and theorems used in this thesis.

Definition 1.1 (Permutation)**.** *If X is a nonempty set, a permutation is the bijective mapping $f : X \to X$.*

Definition 1.2 (Symmetric Group)**.** *The symmetric group, denoted S_n is the the set of all permutions of the nonempty set $X = \{1, 2, ..., n\}$. S_n is a group of order $n!$ on n letters.*

Definition 1.3 (Disjoint)**.** *Two permutations α, $\beta \in S_X$ are disjoint if every x moved by one is fixed by the other. In symbols,*

$$\text{if } \alpha(a) \neq, \text{ then } \beta(a) = a, \text{ and if } \alpha(b) = b, \text{ then } \beta(b) \neq b.$$

Theorem 1.4. *Every permutation of S_n for $n \geq 2$ is a product of disjoint cycles.*

Theorem 1.5. *Let $\alpha \in S_X$, α is either a cycle or a product of disjoint cycles. [**?**]*

Definition 1.6 (Homomorphism)**.** *Let G and H be groups.The mapping $\phi : G \to H$ is said to be a homomorphism if $\forall\ a, b \in G$*

$$\phi(ab) = \phi(a)\phi(b)$$

Definition 1.7 (Isomorphism)**.** *Let G and H be groups. The mapping $\phi : G \to H$ is said to be an isomorphism if the following are true.*

1. ϕ *is a homomorphism*

2. ϕ *is one-to-one*

3. ϕ *is onto*

Definition 1.8 (Conjugate)**.** *Let* G *be a group and* $a \in G$*. For some* $g \in G$, $g^{-1}ag$ *is a conjugate of* a*.*

Definition 1.9 (Simple)**.** *A group* G *is simple if it's only normal subgroups are 1 and* G *itself.*

Definition 1.10 (Normal Subgroup)**.** *Let* H *be a subgroup of* G*. We say* H *is a normal subgroup in* G*, denoted* $H \trianglelefteq G$*, if* $\forall\ g \in G$

$$g^{-1}Hg = H$$

Theorem 1.11 (First Isomorphism Theorem)**.** *Let* $\phi : G \to H$ *be a homomorphism with kernel* K*. Then the following are true.*

1. $K \trianglelefteq G$

2. $G/K \cong im\ \phi$

Theorem 1.12 (Second Isomorphism Theorem)**.** *Let* H *and* K *be subgroups of* G*, where* H *is normal. Then the following are true.*

1. $H \cap K \trianglelefteq K$

2. $K/(H \cap K) \cong HK/H$

Definition 1.13 (Order)**.** *Let* G *be a group. The order of* G *is the number of elements in* G*, denoted* $|G|$*.*

Definition 1.14 (Semi-Direct Product)**.** *The semi-direct product is* $p^{*n} : N$ *where* p^{*n} *denotes a free product of* n *copies of the cyclic group of order* p *and* N *is a transitive permutation group of degree* n *which permutes the* n *generators of the cyclic groups by conjugation. [**?**]*

Definition 1.15 (Progenitor). *A member of a family of infinite groups, the members of which include among their homomorphic images all the non-abelian simple groups, is called a progenitor. [**?**]*

Definition 1.16 (Automorphism). *An isomorphism $\phi : G \to G$ is called an automorphism of G.*

Definition 1.17 (Complement). *Let Q be a group and normal in G. The supgroup is a complement of K in G if*

1. $G = QK$
2. $K \cap Q = 1$

Theorem 1.18 (Feit-Thompson Theorem). *Every simple group is generated by involutions (elements of order 2).*

Chapter 2

Double Coset Enumeration

2.1 Preliminaries

The semi-direct product $p^{*n} : N$ is a progenitor of infinite order. We add relations to a progenitor to factor the group to find finite homomorphic images of $p^{*n} : N$. Double Coset Enumeration is the process used to construct these images. This chapter will provide several examples of how we apply the process to our progenitors.

Definition 2.1 (Right Coset). *Let K be a subgroup of G and $h \in G$. A right coset of K in G is the subset of G defined as*

$$Kh = \{kh | k \in K\},$$

where h is a representative of Kh.

Definition 2.2 (Double Coset). *Let H and K be subgroups of G. Then we define a double coset of G as*

$$HgK = \{Hgk | k \in K, g \in G\}.$$

We note that every double coset is composed of single cosets and G is the union of all double cosets of G.

Definition 2.3 (Stabiliser). *The stabiliser of w in N, where w in a word of t_i's is given by*

$$N^w = \{n \in N | w^n = w\}.$$

Definition 2.4 (Coset Stabiliser)**.** *The coset stabiliser of w in Nw, where w is a word of t_i's is given by*

$$N^{(w)} = \{n \in N | Nw^n = Nw\}.$$

We note that $N^w \leq N^{(w)}$.

Definition 2.5 (G-Orbit)**.** *Let $x \in X$. The set $X^G = \{x^g | g \in G\}$ is the G-orbit.*

To complete the double coset enumeration we seek all the double cosets in G since G is equal to the union of all double cosets in G. The size of each double coset is based on the number of unique single cosets contained in the double coset. The number of single cosets contained in a double coset can be found by $\frac{|N|}{|N^{(w)}|}$, where $N^{(w)} = \{n \in N | Nw^n = Nw\}$ is the coset stabiliser. Then we selct a representative t_i from each orbit of the double coset Nw on $\{1, 2, ..., n\}$ and see which double coset contains Nwt_i. Once the set of right cosets is closed under right multiplication we have no new double cosets and the process of double coset enumeration has been completed. We can then form our Cayley Diagram for the progenitor.

2.2 Double Coset Enumeration of A_5 over S_3

Consider the group $\mathcal{G} = \frac{2^{*3}:S_3}{[(312)t_3]^5,[(31)t_3]^5} \cong A_5$, where $\mathcal{G}$ is the homomorphic image of the infinite semi-direct product of the progenitor $2^{*3} : S_3$ factored by the relations $[(312)t_3]^5, [(31)t_3]^5$. We have $2^{*3} = \langle t_1 \rangle * \langle t_2 \rangle * \langle t_3 \rangle$, where the $t_i s$ are of order 2. We note that $S_3 = \langle (1, 2, 3), (1, 2) \rangle$. Let $x = (1, 2, 3)$ and $y = (1, 2)$.

First we must study our relation $[(3, 1, 2)t_3]^5 = e$.

Let $\pi = (312)$, then

$$\begin{aligned}
\pi t_3\pi t_3\pi t_3\pi t_3\pi t_3 &= e && (2.1)\\
\pi t_3\pi t_3\pi t_3\pi\pi\pi^{-1}t_3\pi t_3 &= e && (2.2)\\
\pi t_3\pi t_3\pi t_3\pi^2(t_3)^{\pi}t_3 &= e && (2.3)\\
\pi t_3\pi t_3\pi\pi^2\pi^{-2}t_3\pi^2(t_3)^{\pi}t_3 &= e && (2.4)\\
\pi t_3\pi t_3\pi^3 t_3^{\pi^2}(t_3)^{\pi}t_3 &= e && (2.5)\\
\pi t_3\pi\pi^3\pi^{-3}t_3\pi^3 t_3^{\pi^2}(t_3)^{\pi}t_3 &= e && (2.6)\\
\pi t_3\pi^4 t_3^{\pi^3}t_3^{\pi^2}(t_3)^{\pi}t_3 &= e && (2.7)\\
\pi\pi^4\pi^{-4}t_3\pi^4 t_3^{\pi^3}t_3^{\pi^2}(t_3)^{\pi}t_3 &= e && (2.8)\\
\pi^5 t_3^{\pi^4}t_3^{\pi^3}t_3^{\pi^2}(t_3)^{\pi}t_3 &= e && (2.9)\\
& && (2.10)
\end{aligned}$$

We know $\pi = (3,1,2)$.

$\Rightarrow \pi^2 = (3,1,2)(3,1,2) = (3,2,1)$

$\Rightarrow \pi^3 = e$

$\Rightarrow \pi^4 = (3,1,2)$

$\Rightarrow \pi^5 = (3,2,1)$

Then our relation is $(3,2,1)t_3^{\pi^4}t_3^{\pi^3}t_3^{\pi^2}(t_3)^{\pi}t_3 = e$.

$\Rightarrow (3,2,1){t_3}^{(3,1,2)}{t_3}^{e}t_3^{(3,2,1)}t_3^{(3,1,2)}t_3 = e$

$\Rightarrow (3,2,1)t_1t_3t_2t_1t_3 = e$

$\Rightarrow (3,2,1)t_1t_3t_2 = t_3t_1$

Thus $(3,2,1)t_1t_3t_2 = t_3t_1$ is our first relation.

Similarly we will study our second relation $[(3,1)t_3]^5$. $[(3,1)t_3]^5 = e$.
Let $\pi = (3,1,2)$, then following the same process as above we find the relation to be $\pi^5 t_3^{\pi^4}t_3^{\pi^3}t_3^{\pi^2}(t_3)^{\pi}t_3 = e$.

However, now we have $\pi = (3,1)$.

$\Rightarrow \pi^2 = e$

$\Rightarrow \pi^3 = (3,1)$

$\Rightarrow \pi^4 = e$

$\Rightarrow \pi^5 = (3,1)$

Then our relation is $(3,1)t_3^{\pi^4}t_3^{\pi^3}t_3^{\pi^2}(t_3)^{\pi}t_3 = e$.

$\Rightarrow (3,1)t_3{}^{e}t_3{}^{(3,1)}t_3^{e}t_3^{(3,1)}t_3 = e$

$\Rightarrow (3,1)t_3t_1t_3t_1t_3 = e$

$\Rightarrow (3,1)t_3t_1t_3 = t_3t_1$

Thus $(3,1)t_3t_1t_3 = t_3t_1$ is our second relation.

Now we will perform the double coset enumeration of $\mathcal{G}$ over N. Let us first consider the double coset containing the identity e. We note that $NeN = \{Nen|n \in N\} = \{Nn|n \in N\} = \{N\}$. Let [$*$] represent the double coset NeN which contains the single coset N. The orbit of N on $\{1,2,3\}$ is $\{1,2,3\}$. We select a representative from our orbit, say 3, and determine which double coset contains Nt_3.

Now $Nt_3 \in Nt_3N$ which is a new double coset. Let [3] denote the new double coset. Now we must consider the coset stabilizer of $N^{(3)}$. $N^{(3)} = \langle(1,3),(1,2)\rangle$. The number of single cosets in [3] can be found by $\frac{|N|}{|N^{(3)}|} = \frac{6}{2} = 3$. The single cosets found within the double coset [3] are $\{Nt_1, Nt_2, Nt_3\}$. The orbits of $N^{(3)}$ on $\{1,2,3\}$ are $\{1,2\}$ and $\{3\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1,2\}$ and $3 \in \{3\}$. We note that $Nt_3t_3 = Nt_3^2 = Ne = N \in [*]$ and Nt_3t_1 belongs to a new double coset. Let [31] denote the new double coset to which Nt_3t_1 belongs. Since we know 1, and 2 are in the same orbit of $N^{(3)}$ on $\{1,2,3\}$, $Nt_3t_2 \in Nt_3t_1 = [31]$.

Now we must consider the coset stabilizer of $N^{(31)}$. $N^{(31)} = \langle Id(N)\rangle$. The number of single cosets in [31] can be found by $\frac{|N|}{|N^{(31)}|} = \frac{6}{1} = 6$. The single cosets found within the double coset [31] are $\{Nt_1t_2, Nt_2t_1, Nt_1t_3, Nt_3t_1, Nt_3t_2, Nt_2t_3\}$. The orbits of $N^{(31)}$ on $\{1,2,3\}$ are $\{1\}$, $\{2\}$, and $\{3\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, and $3 \in \{3\}$. We note that $Nt_3t_1t_1 = Nt_3t_1^2 = Nt_3 \in [3]$. Now $Nt_3t_1t_2 = Nt_3t_1 \in [31]$ by our first relation and $Nt_3t_1t_3 = Nt_3t_1 \in [31]$ by our second relation. Thus we have no new double cosets and we can now construct the following Cayley Diagram.

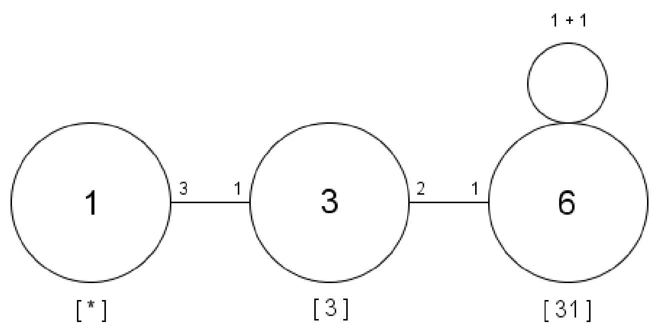

Figure 2.1: Cayley Diagram of A_5 over S_3

2.3 Double Coset Enumeration of $PGL_2(7)$ over S_4

Consider the group $\mathcal{G} = \frac{2^{*4}:S_4}{[t_1t_2t_1t_2=(3,4)]} \cong PGL_2(7)$, where $\mathcal{G}$ is the homomorphic image of the infinite semi-direct product of the progenitor $2^{*4} : S_4$ factored by the relation $[t_1t_2t_1t_2 = (3,4)]$. We have $2^{*4} = \langle t_1 \rangle * \langle t_2 \rangle * \langle t_3 \rangle * \langle t_4 \rangle$, where the $t_i s$ are of order 2. We note that $S_4 = \langle (1,2,3,4), (1,2) \rangle$. Let $x = (1,2,3,4)$ and $y = (1,2)$.

Now we will perform the double coset enumeration of $\mathcal{G}$ over N. Let us first consider the double coset containing the identity e. We note that $NeN = \{Nen | n \in N\} = \{Nn | n \in N\} = \{N\}$. Let $[*]$ represent the double coset NeN which contains the single coset N. The orbit of N on $\{1,2,3,4\}$ is $\{1,2,3,4\}$. We select a representative from our orbit, say 1, and determine which double coset contains Nt_1.

Now $Nt_1 \in Nt_1N$ which is a new double coset. Let [1] denote the new double coset. Now we must consider the coset stabilizer of $N^{(1)}$. $N^{(1)} = \langle (2,3,4), (2,3) \rangle \cong S_3$. The number of single cosets in [1] can be found by $\frac{|N|}{|N^{(1)}|} = \frac{24}{6} = 4$. The single cosets found within the double coset [1] are $\{Nt_1, Nt_2, Nt_3, Nt_4\}$. The orbits of $N^{(1)}$ on $\{1,2,3,4\}$ are $\{1\}$ and $\{2,3,4\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$ and $2 \in \{2,3,4\}$. We note that $Nt_1t_1 = Nt_1^2 = Ne = N \in [*]$ and Nt_1t_2 belongs to a new double coset. Let [12] denote the new double coset to which Nt_1t_2 belongs. Since we know 2, 3, and 4 are in the same orbit of $N^{(1)}$ on $\{1,2,3,4\}$, $Nt_1t_3 \in Nt_1t_2 = [12]$ and $Nt_1t_4 \in Nt_1t_2 = [12]$.

Now we must consider the coset stabilizer of $N^{(12)}$. $N^{(12)} = \langle (3,4) \rangle$. The number of single cosets in [12] can be found by $\frac{|N|}{|N^{(12)}|} = \frac{24}{4} = 6$. The single cosets found within the double coset [12] are $\{Nt_1t_2, Nt_2t_3, Nt_2t_4, Nt_3t_1, Nt_3t_4, Nt_4t_1\}$. The orbits of $N^{(12)}$ on $\{1,2,3,4\}$ are $\{1,2\}$ and $\{3,4\}$. Now we select a representative t_i from each orbit and determine the double coset containg each t_i. Let us select $2 \in \{1,2\}$ and $3 \in \{3,4\}$. We note that $Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$ and $Nt_1t_2t_3$ belongs to a new double coset. Let [123] denote the new double coset to which $Nt_1t_2t_3$ belongs. Since 1 and 2 are in the same orbit of $N^{(12)}$ on $\{1,2,3,4\}$, $Nt_1t_2t_1 \in Nt_1 = [1]$. Since 3 and 4 are in the same orbit of $N^{(12)}$ on $\{1,2,3,4\}$, $Nt_1t_2t_4 \in Nt_1t_2t_3 = [123]$.

Now we must consider the coset stabilizer of $N^{(123)}$. $N^{(123)} = \langle(1,2),(2,3),(1,2)(3,4)\rangle$. The number of single cosets in [123] can be found by $\frac{|N|}{|N^{(123)}|} = \frac{24}{8} = 3$. The single cosets found within the double coset [123] are $\{Nt_1t_2t_3, Nt_2t_3t_4, Nt_2t_4t_1\}$. The orbit of $N^{(123)}$ on $\{1,2,3,4\}$ is $\{1,2,3,4\}$. Now we select a representative t_i from the orbit and determine the double coset containing it. Let us select $3 \in \{1,2,3,4\}$. We note that $Nt_1t_2t_3t_3 = Nt_1t_2t_3^2 = Nt_1t_2 \in [12]$. Since 1, 2, 3 and 4 are in the same orbit of $N^{(123)}$ on $\{1,2,3,4\}$, $Nt_1t_2t_3t_1 \in Nt_1t_2 = [12]$ and $Nt_1t_2t_3t_2 \in Nt_1t_2 = [12]$ and $Nt_1t_2t_3t_4 \in Nt_1t_2 = [12]$. So we have no new double cosets and we can now construct the following Cayley Diagram.

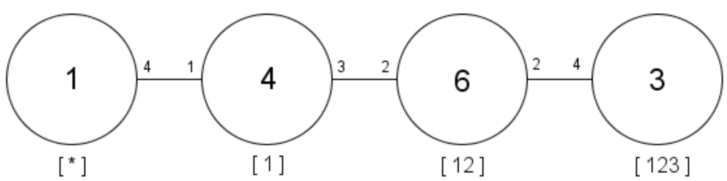

Figure 2.2: Cayley Diagram of $PGL_2(7)$ over S_4

2.4 Double Coset Enumeration of $[3 \times A_5] : 2$ over D_{10}

Consider the group $\mathcal{G} = \frac{2^{*10}:D_{10}}{[(x^2ytt^x)^3,(xt^x)^3]} \cong [3 \times A_5] : 2$, where $\mathcal{G}$ is the homomorphic image of the infinite semi-direct product of the progenitor $2^{*10} : D_{10}$ factored by the relations $[x^2ytt^x]^3$, $[xt^x]^3$. We have $2^{*10} = \langle t_1\rangle * \langle t_2\rangle * \langle t_3\rangle * \langle t_4\rangle * \langle t_5\rangle * \langle t_6\rangle * \langle t_7\rangle *$

$\langle t_8\rangle * \langle t_9\rangle * \langle t_{10}\rangle$, where the t_is are of order 2. We note that $D_{10} = \langle(1,2,3,4),(1,2)\rangle$. Let $x = (1,2,3,4,5,6,7,8,9,10)$ and $y = (2,10)(3,9)(4,8)(5,7)$.

First we must study our relation $[x^2ytt^x]^3 = e$.

Let $\pi = x^2y = (1,9)(2,8)(3,7)(4,6)$, then

$$(\pi tt^x)^3 = e \quad (2.11)$$

$$(\pi t_1t_2)^3 = e \quad (2.12)$$

$$\pi t_1t_2\pi t_1t_2\pi t_1t_2 = e \quad (2.13)$$

$$\pi t_1t_2\pi^2 t_1^{\pi}t_2^{\pi}t_1t_2 = e \quad (2.14)$$

$$\pi^3 t_1^{\pi^2}t_2^{\pi^2}t_1^{\pi}t_2^{\pi}t_1t_2 = e \quad (2.15)$$

$$(2.16)$$

We know $\pi = (1,9)(2,8)(3,7)(4,6)$.

$\Rightarrow \pi^2 = e$

$\Rightarrow \pi^3 = (1,9)(2,8)(3,7)(4,6)$

Then our relation is $(1,9)(2,8)(3,7)(4,6)t_1^{\pi^2}t_2^{\pi^2}t_1^{\pi}t_2^{\pi}t_1t_2 = e$.

$\Rightarrow (1,9)(2,8)(3,7)(4,6)t_1^e t_2^e t_1^{(1,9)(2,8)(3,7)(4,6)}t_2^{(1,9)(2,8)(3,7)(4,6)}t_1t_2 = e$

$\Rightarrow (1,9)(2,8)(3,7)(4,6)t_1t_2t_9t_8t_1t_2 = e$

$\Rightarrow (1,9)(2,8)(3,7)(4,6)t_1t_2t_9 = t_2t_1t_8$

Thus $(1,9)(2,8)(3,7)(4,6)t_1t_2t_9 = t_2t_1t_8$ is our first relation.

Similarly we will study our second relation $[xt^x]^3$. $[xt^x]^3 = e$.

Let $\pi = (1,2,3,4,5,6,7,8,9,10)$, then

$$(\pi t^x)^3 = e \quad (2.17)$$

$$(\pi t_2)^3 = e \quad (2.18)$$

$$\pi t_2\pi t_2\pi t_2 = e \quad (2.19)$$

$$\pi t_2\pi^2 t_2^{\pi}t_2 = e \quad (2.20)$$

$$\pi^3 t_2^{\pi^2}t_2^{\pi}t_2 = e \quad (2.21)$$

Now we have $\pi = (1,2,3,4,5,6,7,8,9,10)$.

$\Rightarrow \pi^2 = (1,3,5,7,9)(2,4,6,8,10)$

$\Rightarrow \pi^3 = (1,4,7,10,3,6,9,2,5,8)$

Then our relation is $(1,4,7,10,3,6,9,2,5,8)t_2^{\pi^2}t_2^{\pi}t_2 = e$.

$\Rightarrow (1,4,7,10,3,6,9,2,5,8){t_2}^{(1,3,5,7,9)(2,4,6,8,10)}t_2^{(1,2,3,4,5,6,7,8,9,10)}t_2 = e$

$\Rightarrow (1,4,7,10,3,6,9,2,5,8)t_4t_3t_2 = e$

$\Rightarrow (1,4,7,10,3,6,9,2,5,8)t_4 = t_2t_3$

Thus $(1,4,7,10,3,6,9,2,5,8)t_4 = t_2t_3$ is our second relation.

Now we will perform the double coset enumeration of $\mathcal{G}$ over N. Let us first consider the double coset containing the identity e. We note that $NeN = \{Nen|n \in N\} = \{Nn|n \in N\} = \{N\}$. Let $[*]$ represent the double coset NeN which contains the single coset N. The orbit of N on $\{1,2,3,4,5,6,7,8,9,10\}$ is $\{1,2,3,4,5,6,7,8,9,10\}$. We select a representative from our orbit, say 1, and determine which double coset contains Nt_1.

Now $Nt_1 \in Nt_1N$ which is a new double coset. Let [1] denote the new double coset. Now we must consider the coset stabilizer of $N^{(1)}$. $N^{(1)} = \{e,(2,10)(3,9)(4,8)(5,7)\}$. The number of single cosets in [1] can be found by $\frac{|N|}{|N^{(1)}|} = \frac{20}{2} = 10$. The single cosets found within the double coset [1] are $\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}\}$. The orbits of $N^{(1)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}$, $\{6\}$, $\{2,10\}$, $\{3,9\}$, $\{4,8\}$, and $\{5,7\}$. Now we select a representative t_i from each orbit and determine the double coset containg each t_i. Let us select $1 \in \{1\}$ and $6 \in \{6\}$. We note that $Nt_1t_1 = Nt_1^2 = Ne = N \in [*]$ and Nt_1t_6 belongs to a new double coset. Let [16] denote the new double coset to which Nt_1t_6 belongs. Similarly we find the double cosets Nt_1t_2, Nt_1t_3, Nt_1t_4, and Nt_1t_5. Upon further examination, through MAGMA we find that $Nt_1t_2 = Nt_1$, $Nt_1t_4 = Nt_1$, and $Nt_1t_5 = Nt_1t_3$, which we will denote [13].

Now we must consider the coset stabilizer of $N^{(16)}$. $N^{(16)} = \{e, (2,10)(3,9)(4,8)(5,7), (1,3,5,7,9)(2,4,6,8,10), (1,3)(4,10)(5,9)(6,8), (1,9)(2,8)(3,7)(4,6), (1,9,7,5,3)(2,10,8,6,4), (1,5,9,3,7)(2,6,10,4,8), (15)(2,4)(6,10)(7,9), (1,7,3,9,5)(2,8,4,10,6), (1,7)(2,6)(3,5)(8,10)\}$. The number of single cosets in [16] can be found by $\frac{|N|}{|N^{(16)}|} = \frac{20}{10} = 2$. The orbits of $N^{(16)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1,3,5,7,9\}$ and $\{2,4,6,8,10\}$. Now we select a representative t_i from each orbit and determine the double coset containg each t_i. Let us select $1 \in \{1,3,5,7,9\}$

and $6 \in \{2,4,6,8,10\}$. We note that $Nt_1t_6t_6 = Nt_1t_6^2 = Nt_1 \in [1]$ and $Nt_1t_6t_1$ belongs to a new double coset. Let [161] denote the new double coset to which $Nt_1t_6t_1$ belongs. Since 2, 4, 6, 8, and 10 are in the same orbit of $N^{(16)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_6t_2$, $Nt_1t_6t_4$, $Nt_1t_6t_8$, and $Nt_1t_6t_{10} \in Nt_1 = [1]$. Since 1, 3, 5, 7, and 9 are in the same orbit of $N^{(16)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_6t_3$, $Nt_1t_6t_5$, $Nt_1t_6t_7$, and $Nt_1t_6t_9 \in Nt_1t_6t_1 = [161]$. Upon further examination, through MAGMA we find that $Nt_1t_6t_1 = Nt_1t_3$, which we will denote [13].

Now we must consider the coset stabilizer of $N^{(13)}$. $N^{(13)} = \{e, (2,10)(3,9)$ $(4,8)(5,7), (1,6)(2,7)(3,8)(4,9)(5,10), (1,6)(2,5)(3,4)(7,10)(8,9)\}$. The number of single cosets in [13] can be found by $\frac{|N|}{|N^{(13)}|} = \frac{20}{4} = 5$. The orbits of $N^{(13)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1,6\}$, $\{2,5,7,10\}$ and $\{3,4,8,9\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1,6\}$, $2 \in \{2,5,7,10\}$ and $3 \in \{3,4,8,9\}$. We note that $Nt_1t_3t_3 = Nt_1t_3^2 = Nt_1 \in [1]$. Through MAGMA we find that $Nt_1t_3t_1 = Nt_1t_6$ and $Nt_1t_3t_2 = Nt_1$. Since 1 and 6 are in the same orbit of $N^{(13)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_3t_6 \in Nt_1t_6 = [16]$. Similarly, $Nt_1t_3t_5, Nt_1t_3t_7, Nt_1t_3t_{10} \in Nt_1 = [1]$, and $Nt_1t_3t_4, Nt_1t_3t_8, Nt_1t_3t_9 \in Nt_1t_2 = [12]$. So we have no new double cosets and we can now construct the Cayley Diagram as seen in figure **??**.

2.5 Double Coset Enumeration of $3 : [A_6 : 2]$ over D_{10}

Consider the group $\mathcal{G} = \frac{2^{*10}:D_{10}}{[(x^2ytt^x)^3,(xt^x)^5,(tt^xt^{x^3})^8]} \cong 3 : [A_6 : 2]$, where $\mathcal{G}$ is the homomorphic image of the infinite semi-direct product of the progenitor $2^{*10} : D_{10}$ factored by the relations $[x^2ytt^x]^3$, $[xt^x]^5$, $[tt^xt^{x^3}]^8$. We have $2^{*10} = \langle t_1\rangle * \langle t_2\rangle * \langle t_3\rangle * \langle t_4\rangle * \langle t_5\rangle * \langle t_6\rangle * \langle t_7\rangle * \langle t_8\rangle * \langle t_9\rangle * \langle t_{10}\rangle$, where the t_is are of order 2. We note that $D_{10} = \langle (1,2,3,4),(1,2)\rangle$. Let $x = (1,2,3,4,5,6,7,8,9,10)$ and $y = (2,10)(3,9)(4,8)(5,7)$.

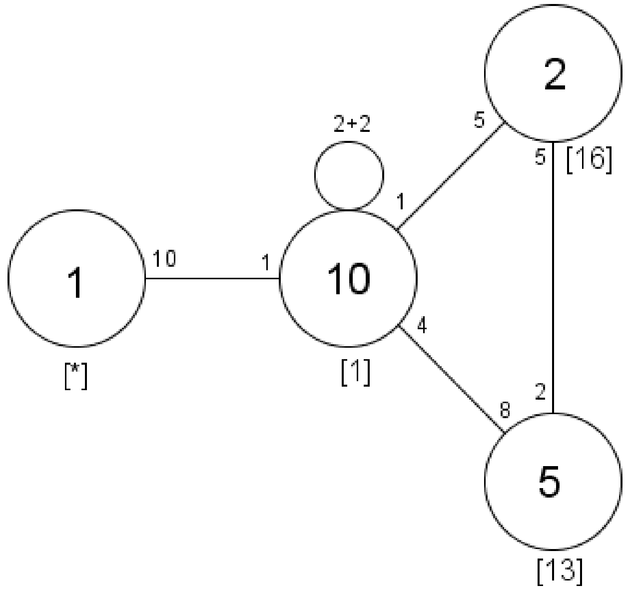

Figure 2.3: Cayley Diagram of $[3 \times A_5] : 2$ over D_{10}

First we must study our relation $[x^2ytt^x]^3 = e$.
Let $\pi = x^2y = (1,9)(2,8)(3,7)(4,6)$, then

$$(\pi tt^x)^3 = e \quad (2.22)$$

$$(\pi t_1 t_2)^3 = e \quad (2.23)$$

$$\pi t_1 t_2 \pi t_1 t_2 \pi t_1 t_2 = e \quad (2.24)$$

$$\pi t_1 t_2 \pi^2 t_1^{\pi} t_2^{\pi} t_1 t_2 = e \quad (2.25)$$

$$\pi^3 t_1^{\pi^2} t_2^{\pi^2} t_1^{\pi} t_2^{\pi} t_1 t_2 = e \quad (2.26)$$

$$(2.27)$$

We know $\pi = (1,9)(2,8)(3,7)(4,6)$.
$\Rightarrow \pi^2 = e$
$\Rightarrow \pi^3 = (1,9)(2,8)(3,7)(4,6)$
Then our relation is $(1,9)(2,8)(3,7)(4,6)t_1^{\pi^2} t_2^{\pi^2} t_1^{\pi} t_2^{\pi} t_1 t_2 = e$.
$\Rightarrow (1,9)(2,8)(3,7)(4,6)t_1^e t_2^e t_1^{(1,9)(2,8)(3,7)(4,6)} t_2^{(1,9)(2,8)(3,7)(4,6)} t_1 t_2 = e$

$\Rightarrow (1,9)(2,8)(3,7)(4,6)t_1t_2t_9t_8t_1t_2 = e$
$\Rightarrow (1,9)(2,8)(3,7)(4,6)t_1t_2t_9 = t_2t_1t_8$
Thus $(1,9)(2,8)(3,7)(4,6)t_1t_2t_9 = t_2t_1t_8$ is our first relation.

Next we will study our second relation $[xt^x]^5 = e$.
We have $x = (1,2,3,4,5,6,7,8,9,10)$, then

$$(xt^x)^5 = e \tag{2.28}$$
$$(xt_2)^5 = e \tag{2.29}$$
$$xt_2xt_2xt_2xt_2xt_2 = e \tag{2.30}$$
$$xt_2xt_2xt_2x^2t_2^xt_2 = e \tag{2.31}$$
$$xt_2xt_2x^3t_2^{x^2}t_2^xt_2 = e \tag{2.32}$$
$$xt_2x^4t_2^{x^3}t_2^{x^2}t_2^xt_2 = e \tag{2.33}$$
$$x^5t_2^{x^4}t_2^{x^3}t_2^{x^2}t_2^xt_2 = e \tag{2.34}$$
$$\tag{2.35}$$

Now we have $x = (1,2,3,4,5,6,7,8,9,10)$.
$\Rightarrow x^2 = (1,3,5,7,9)(2,4,6,8,10)$
$\Rightarrow x^3 = (1,4,7,10,3,6,9,2,5,8)$
$\Rightarrow x^4 = (1,5,9,3,7)(2,6,10,4,8)$
$\Rightarrow x^5 = (1,6)(2,7)(3,8)(4,9)(5,10)$
Then our relation is $(1,6)(2,7)(3,8)(4,9)(5,10)t_2^{x^4}t_2^{x^3}t_2^{x^2}t_2^xt_2 = e$.
$\Rightarrow (1,6)(2,7)(3,8)(4,9)(5,10)t_2^{x^4}t_2^{x^3}t_2^{x^2}t_2^xt_2 = e$
$\Rightarrow (1,6)(2,7)(3,8)(4,9)(5,10)t_8t_5t_4t_3t_2 = e$
$\Rightarrow (1,6)(2,7)(3,8)(4,9)(5,10)t_8t_5 = t_2t_3t_4$
Thus $(1,6)(2,7)(3,8)(4,9)(5,10)t_8t_5 = t_2t_3t_4$ is our second relation.

Our final relation is $[tt^xt^{x^3}]^8$. $[tt^xt^{x^3}]^8 = e$
We have $x = (1,2,3,4,5,6,7,8,9,10)$.
$\Rightarrow x^3 = (1,4,7,10,3,6,9,2,5,8)$
$\Rightarrow [tt^xt^{x^3}]^8 = [t_1t_2t_4]^8$

$\Rightarrow [t_1t_2t_4]^8 = t_1t_2t_4t_1t_2t_4t_1t_2t_4t_1t_2t_4t_1t_2t_4t_1t_2t_4t_1t_2t_4t_1t_2t_4$

Now we will perform the double coset enumeration of $\mathcal{G}$ over N. Let us first consider the double coset containing the identity e. We note that $NeN = \{Nen|n \in N\} = \{Nn|n \in N\} = \{N\}$. Let $[*]$ represent the double coset NeN which contains the single coset N. The orbit of N on $\{1,2,3,4,5,6,7,8,9,10\}$ is $\{1,2,3,4,5,6,7,8,9,10\}$. We select a representative from our orbit, say 1, and determine which double coset contains Nt_1.

Now $Nt_1 \in Nt_1N$ which is a new double coset. Let [1] denote the new double coset. Now we must consider the coset stabilizer of $N^{(1)}$. $N^{(1)} = \{e, (2,10)(3,9)(4,8)(5,7)\}$. The number of single cosets in [1] can be found by $\frac{|N|}{|N^{(1)}|} = \frac{20}{2} = 10$. The single cosets found within the double coset [1] are $\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}\}$. The orbits of $N^{(1)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}$, $\{6\}$, $\{2,10\}$, $\{3,9\}$, $\{4,8\}$, and $\{5,7\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$ and $6 \in \{6\}$. We note that $Nt_1t_1 = Nt_1^2 = Ne = N \in [*]$ and Nt_1t_6 belongs to a new double coset. Let [16] denote the new double coset to which Nt_1t_6 belongs. Similarly we find the new double cosets Nt_1t_2, Nt_1t_3, Nt_1t_4, and Nt_1t_5, which we will denote [12], [13], [14], and [15] respectively.

Now we must consider the coset stabilizer of $N^{(16)}$. $N^{(16)} = \{e, (2,10)(3,9)(4,8)(5,7), (1,3,5,7,9)(2,4,6,8,10), (1,3)(4,10)(5,9)(6,8), (1,9)(2,8)(3,7)(4,6), (1,9,7,5,3)(2,10,8,6,4), (1,5,9,3,7)(2,6,10,4,8), (15)(2,4)(6,10)(7,9), (1,7,3,9,5)(2,8,4,10,6), (1,7)(2,6)(3,5)(8,10)\}$. The number of single cosets in [16] can be found by $\frac{|N|}{|N^{(16)}|} = \frac{20}{10} = 2$. The orbits of $N^{(16)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1,3,5,7,9\}$ and $\{2,4,6,8,10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1,3,5,7,9\}$ and $6 \in \{2,4,6,8,10\}$. We note that $Nt_1t_6t_6 = Nt_1t_6^2 = Nt_1 \in [1]$ and $Nt_1t_6t_1$ belongs to a new double coset. Let [161] denote the new double coset to which $Nt_1t_6t_1$ belongs. Since 2, 4, 6, 8, and 10 are in the same orbit of $N^{(16)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_6t_2$, $Nt_1t_6t_4$, $Nt_1t_6t_8$, and $Nt_1t_6t_{10} \in Nt_1 = [1]$. Since 1, 3, 5, 7, and 9 are in the same orbit of $N^{(16)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_6t_3$, $Nt_1t_6t_5$, $Nt_1t_6t_7$, and $Nt_1t_6t_9 \in Nt_1t_6t_1 = [161]$.

Now we must consider the coset stabilizer of $N^{(13)}$. $N^{(13)} = \{e, (1,10)(2,9)$

$(3,8)(4,7)(5,6)\}$. The number of single cosets in [13] can be found by $\frac{|N|}{|N^{(13)}|} = \frac{20}{2} = 10$. The orbits of $N^{(13)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1,10\}$, $\{2,9\}$, $\{3,8\}$, $\{4,7\}$, and $\{5,6\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1,10\}$, $2 \in \{2,9\}$, $3 \in \{3,8\}$, $4 \in \{4,7\}$, and $5 \in \{5,6\}$. We note that $Nt_1t_3t_3 = Nt_1t_3^2 = Nt_1 \in [1]$. We note that $Nt_1t_3t_2$ and $Nt_1t_3t_4$ belong to new double cosets denoted [132] and [134] respectively. Through MAGMA we find that $Nt_1t_3t_1 = Nt_1t_3$ and $Nt_1t_3t_5 = Nt_1t_4$. Since 1 and 10 are in the same orbit of $N^{(13)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_3t_{10} \in Nt_1t_3 = [13]$. Similarly, $Nt_1t_3t_9 \in [132], Nt_1t_3t_8 \in [13], Nt_1t_3t_7 \in [134]$, and $Nt_1t_3t_6 \in [14]$.

Now we must consider the coset stabilizer of $N^{(15)}$. $N^{(15)} = \{e, (1,4)(2,3)(5,10)(6,9)(7,8)\}$. The number of single cosets in [15] can be found by $\frac{|N|}{|N^{(15)}|} = \frac{20}{2} = 10$. The orbits of $N^{(15)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1,4\}$, $\{2,3\}, \{5,10\}, \{6,9\}$ and $\{7,8\}$. Now we select a representative t_i from each orbit and determine the double coset conaing each t_i. Let us select $1 \in \{1,4\}$, $2 \in \{2,3\}$, $5 \in \{5,10\}$, $6 \in \{6,9\}$, and $7 \in \{7,8\}$. We note that $Nt_1t_5t_5 = Nt_1t_5^2 = Nt_1 \in [1]$ and $Nt_1t_5t_7$ belongs to a new double coset denoted by [157]. Through MAGMA we see that $Nt_1t_5t_1 = Nt_1t_2 \in [12]$, $Nt_1t_5t_2 = Nt_1t_3t_2 \in [132]$, and $Nt_1t_5t_6 = Nt_1t_5 \in [15]$. Since 4 is in the same orbit of $N^{(15)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_5t_4 \in [12]$. Similarly $Nt_1t_5t_3 \in [132]$, $Nt_1t_5t_10 \in [1]$, $Nt_1t_5t_9 \in [15]$, $Nt_1t_5t_8 \in [157]$.

Now we must consider the coset stabilizer of $N^{(12)}$. $N^{(12)} = \{e\}$. The number of single cosets in [12] can be found by $\frac{|N|}{|N^{(12)}|} = \frac{20}{1} = 20$. The orbits of $N^{(12)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}, 7 \in \{7\}, 8 \in \{8\}, 9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$. We see that $Nt_1t_2t_1$, $Nt_1t_2t_3$, $Nt_1t_2t_4$, $Nt_1t_2t_5$, $Nt_1t_2t_6$, $Nt_1t_2t_7$, $Nt_1t_2t_8$, $Nt_1t_2t_9$, and $Nt_1t_2t_{10}$ belong to new double cosets. Through MAGMA we find that $Nt_1t_2t_1 = Nt_1t_2 \in [12]$, $Nt_1t_2t_3 = Nt_1t_2 \in [12]$, $Nt_1t_2t_4 = Nt_1t_5t_7 \in [157]$, $Nt_1t_2t_5 = Nt_1t_3t_4 \in [134]$, $Nt_1t_2t_6 = Nt_1t_5 \in [15]$, $Nt_1t_2t_7 = Nt_1t_6t_1 \in [161]$, $Nt_1t_2t_8 = Nt_1t_2 \in [12]$, $Nt_1t_2t_9 = Nt_1t_3t_4 \in [134]$, and $Nt_1t_2t_{10} = Nt_1t_3t_2 \in [132]$.

Now we must consider the coset stabilizer of $N^{(14)}$. $N^{(14)} = \{e\}$. The number of single cosets in [14] can be found by $\frac{|N|}{|N^{(14)}|} = \frac{20}{1} = 20$. The orbits of $N^{(14)}$ on

$\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}, 7 \in \{7\}, 8 \in \{8\}, 9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_4t_4 = Nt_1t_4^2 = Nt_1 \in [1]$. We see that $Nt_1t_4t_1$, $Nt_1t_4t_2$, $Nt_1t_4t_3$, $Nt_1t_4t_5$, $Nt_1t_4t_6$, $Nt_1t_4t_7$, $Nt_1t_4t_8$, $Nt_1t_4t_9$, and $Nt_1t_4t_{10}$ belong to new double cosets. Through MAGMA we find that $Nt_1t_4t_1 = Nt_1t_4 \in [14]$, $Nt_1t_4t_2 = Nt_1t_3 \in [13]$, $Nt_1t_4t_3 = Nt_1t_5t_7 \in [157]$, $Nt_1t_4t_5 = Nt_1t_5t_7 \in [157]$, $Nt_1t_4t_6 = Nt_1t_4 \in [14]$, $Nt_1t_4t_7 = Nt_1t_4 \in [14]$, $Nt_1t_4t_8 = Nt_1t_3t_4 \in [134]$, $Nt_1t_4t_9 = Nt_1t_6t_1 \in [161]$, and $Nt_1t_4t_{10} = Nt_1t_3t_2 \in [132]$.

Now we must consider the coset stabilizer of $N^{(132)}$. $N^{(132)} = \{e,(2,10)(3,9)(4,8)(5,7)\}$. The number of single cosets in [132] can be found by $\frac{|N|}{|N^{(132)}|} = \frac{20}{2} = 10$. The orbits of $N^{(132)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}$, $\{6\}$, $\{2,10\}$, $\{3,9\}$, $\{4,8\}$, and $\{5,7\}$. Now we select a representative t_i from each orbit and determine the double coset containg each t_i. Let us select $1 \in \{1\}$, $6 \in \{6\}$, $2 \in \{2,10\}$, $3 \in \{3,9\}$, $4 \in \{4,8\}$, and $5 \in \{5,7\}$. We note that $Nt_1t_3t_2t_2 = Nt_1t_3t_2^2 = Nt_1t_3 \in [13]$. Through MAGMA we find that $Nt_1t_3t_2t_1 = Nt_1t_5t_7 \in [157]$, $Nt_1t_3t_2t_6 = Nt_1t_3t_4 \in [134]$, $Nt_1t_3t_2t_3 = Nt_1t_5 \in [15]$, $Nt_1t_3t_2t_4 = Nt_1t_2 \in [12]$, $Nt_1t_3t_2t_5 = Nt_1t_4 \in [14]$. Since 2 and 10 are in the same orbit of $N^{(132)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_3t_2t_{10} = Nt_1t_5t_7 \in [157]$. Similarly, $Nt_1t_3t_2t_9 \in [15], Nt_1t_3t_2t_8 \in [12], Nt_1t_3t_2t_7 \in [14]$.

Now we must consider the coset stabilizer of $N^{(157)}$. $N^{(157)} = \{e,(1,7)(2,6)(3,5)(8,10)\}$. The number of single cosets in [157] can be found by $\frac{|N|}{|N^{(157)}|} = \frac{20}{2} = 10$. The orbits of $N^{(157)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{4\}$, $\{9\}$, $\{1,7\}$, $\{2,6\}$, $\{3,5\}$, and $\{8,10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $4 \in \{4\}$, $9 \in \{9\}$, $7 \in \{1,7\}$, $2 \in \{2,6\}$, $3 \in \{3,5\}$, and $8 \in \{8,10\}$. We note that $Nt_1t_5t_7t_7 = Nt_1t_5t_7^2 = Nt_1t_5 \in [15]$. Through MAGMA we find that $Nt_1t_5t_7t_4 = Nt_1t_3t_2 \in [132]$, $Nt_1t_5t_7t_9 = Nt_1t_5t_7 \in [157]$, $Nt_1t_5t_7t_2 = Nt_1t_2 \in [12]$, $Nt_1t_5t_7t_3 = Nt_1t_4 \in [14]$, $Nt_1t_5t_7t_8 = Nt_1t_4 \in [14]$. Since 1 and 7 are in the same orbit of $N^{(157)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_5t_7t_1 = Nt_1t_5 \in [15]$. Similarly, $Nt_1t_5t_7t_6 \in [12], Nt_1t_5t_7t_5 \in [14]$, and $Nt_1t_5t_7t_{10} \in [14]$. So we have no new double cosets and we can now construct the following Cayley diagram.

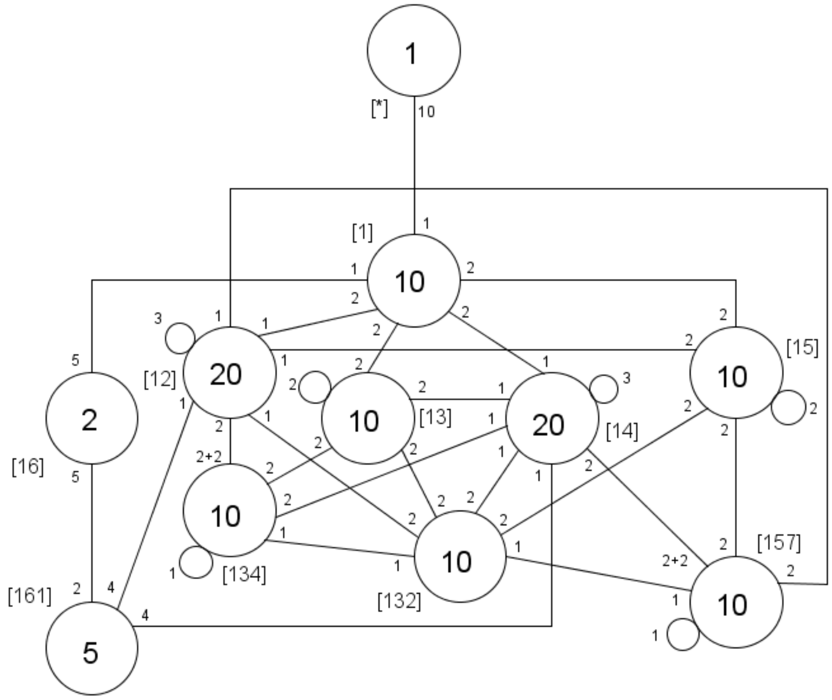

Figure 2.4: Cayley Diagram of $3 : [A_6 : 2]$ over D_{10}

2.6 Double Coset Enumeration of $2 \times [3 \times L_2(11) : 2]$ over D_{10}

Consider the group $\mathcal{G} = \frac{2^{*10}:D_{10}}{[(xt^x)^3,(tt^xt^{x^3})^4]} \cong 2 \times [(3 \times L_2(11)) : 2]$, where $\mathcal{G}$ is the homomorphic image of the infinite semi-direct product of the progenitor $2^{*10} : D_{10}$ factored by the relations $[xt^x]^3$, $[tt^xt^{x^3}]^4$. We have $2^{*10} = \langle t_1 \rangle * \langle t_2 \rangle * \langle t_3 \rangle * \langle t_4 \rangle * \langle t_5 \rangle * \langle t_6 \rangle * \langle t_7 \rangle * \langle t_8 \rangle * \langle t_9 \rangle * \langle t_{10} \rangle$, where the $t_i s$ are of order 2. We note that $D_{10} = \langle (1, 2, 3, 4), (1, 2) \rangle$. Let $x = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ and $y = (2, 10)(3, 9)(4, 8)(5, 7)$.

First we must study our relation $[xt^x]^5 = e$.

We have $x = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$, then

$$\begin{aligned}
(xt^x)^3 &= e &(2.36)\\
(xt_2)^3 &= e &(2.37)\\
xt_2xt_2xt_2 &= e &(2.38)\\
xt_2x^2t_2^xt_2 &= e &(2.39)\\
x^3t_2^{x^2}t_2^xt_2 &= e &(2.40)
\end{aligned}$$

Now we have $x = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.
$\Rightarrow x^2 = (1, 3, 5, 7, 9)(2, 4, 6, 8, 10)$
$\Rightarrow x^3 = (1, 4, 7, 10, 3, 6, 9, 2, 5, 8)$
Then our relation is $(1, 4, 7, 10, 3, 6, 9, 2, 5, 8)t_2^{x^2}t_2^xt_2 = e$.
$\Rightarrow (1, 4, 7, 10, 3, 6, 9, 2, 5, 8){t_2}^{(1,3,5,7,9)(2,4,6,8,10)}t_2^{(1,2,3,4,5,6,7,8,9,10)}t_2 = e$
$\Rightarrow (1, 4, 7, 10, 3, 6, 9, 2, 5, 8)t_4t_3t_2 = e$
$\Rightarrow (1, 4, 7, 10, 3, 6, 9, 2, 5, 8)t_4 = t_2t_3$
Thus $(1, 4, 7, 10, 3, 6, 9, 2, 5, 8)t_4 = t_2t_3$ is our first relation.

Our second relation is $[tt^xt^{x^3}]^4 = e$.
We have $x = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.
$\Rightarrow x^3 = (1, 4, 7, 10, 3, 6, 9, 2, 5, 8)$
$\Rightarrow [tt^xt^{x^3}]^4 = [t_1t_2t_4]^4$
$\Rightarrow [t_1t_2t_4]^4 = t_1t_2t_4t_1t_2t_4t_1t_2t_4t_1t_2t_4$

Now we will perform the double coset enumeration of $\mathcal{G}$ over N. Let us first consider the double coset containing the identity e. We note that $NeN = \{Nen|n \in N\} = \{Nn|n \in N\} = \{N\}$. Let $[*]$ represent the double coset NeN which contains the single coset N. The orbit of N on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. We select a representative from our orbit, say 1, and determine which double coset contains Nt_1.

Now $Nt_1 \in Nt_1N$ which is a new double coset. Let [1] denote the new double coset. Now we must consider the coset stabilizer of $N^{(1)}$. $N^{(1)} = \{e, (2, 10)(3, 9)(4, 8)$

$(5,7)\}$. The number of single cosets in [1] can be found by $\frac{|N|}{|N^{(1)}|} = \frac{20}{2} = 10$. The single cosets found within the double coset [1] are $\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}\}$. The orbits of $N^{(1)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}$, $\{6\}$, $\{2,10\}$, $\{3,9\}$, $\{4,8\}$, and $\{5,7\}$. Now we select a representative t_i from each orbit and determine the double coset conatining each t_i. Let us select $1 \in \{1\}$ and $6 \in \{6\}$. We note that $Nt_1t_1 = Nt_1^2 = Ne = N \in [*]$ and Nt_1t_6 belongs to a new double coset. Let [16] denote the new double coset to which Nt_1t_6 belongs. Similarly we find the new double cosets Nt_1t_3, Nt_1t_4, and Nt_1t_5, which we will denote [13], [14], and [15] respectively. Through MAGMA we find that $Nt_1t_2 = Nt_1 \in [1]$.

Now we must consider the coset stabilizer of $N^{(13)}$. $N^{(13)} = \{e, (1,6)(2,5)(3,4)(7,10)(8,9)\}$. The number of single cosets in [13] can be found by $\frac{|N|}{|N^{(13)}|} = \frac{20}{2} = 10$. The orbits of $N^{(13)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1,6\}$, $\{2,5\}$, $\{3,4\}$, $\{7,10\}$, and $\{8,9\}$. Now we select a representative t_i from each orbit and determine the double coset conatining each t_i. Let us select $1 \in \{1,6\}$, $2 \in \{2,5\}$, $3 \in \{3,4\}$, $7 \in \{7,10\}$, and $8 \in \{8,9\}$. We note that $Nt_1t_3t_3 = Nt_1t_3^2 = Nt_1 \in [1]$. We note that $Nt_1t_3t_1$, $Nt_1t_3t_7$, $Nt_1t_3t_8$ belong to new double cosets denoted [131], [137] and [138] respectively. Through MAGMA we find that $Nt_1t_3t_2 = Nt_1t_4$. Since 6 is in the same orbit of $N^{(13)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_3t_6 \in Nt_1t_3t_1 = [131]$. Similarly, $Nt_1t_3t_5 \in [14], Nt_1t_3t_4 \in [13], Nt_1t_3t_{10} \in [137]$, and $Nt_1t_3t_9 \in [138]$.

Now we must consider the coset stabilizer of $N^{(14)}$. $N^{(14)} = \{e\}$. The number of single cosets in [14] can be found by $\frac{|N|}{|N^{(14)}|} = \frac{20}{1} = 20$. The orbits of $N^{(14)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}, 7 \in \{7\}, 8 \in \{8\}, 9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_4t_4 = Nt_1t_4^2 = Nt_1 \in [1]$. We see that $Nt_1t_4t_1$, $Nt_1t_4t_2$, $Nt_1t_4t_7$, $Nt_1t_4t_9$, $Nt_1t_4t_{10}$ belong to new double cosets. Through MAGMA we find that $Nt_1t_4t_3 = Nt_1t_5 \in [15]$, $Nt_1t_4t_5 = Nt_1t_3 \in [13]$, $Nt_1t_4t_6 = Nt_1t_3t_1 \in [131]$, and $Nt_1t_4t_8 = Nt_1t_4 \in [14]$.

Now we must consider the coset stabilizer of $N^{(15)}$. $N^{(15)} = \{e, (1,3)(4,10)(5,9)(6,8)\}$. The number of single cosets in [15] can be found by $\frac{|N|}{|N^{(15)}|} = \frac{20}{2} = 10$. The orbits of $N^{(15)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{2\}$, $\{7\}, \{1,3\}, \{4,10\}, \{5,9\}$ and $\{6,8\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1,3\}$, $2 \in \{2\}$, $7 \in \{7\}$, $4 \in \{4,10\}$, $5 \in \{5,9\}$, and $6 \in \{6,8\}$. We note that $Nt_1t_5t_5 = Nt_1t_5^2 = Nt_1 \in [1]$ and $Nt_1t_5t_2$ belongs to a new double coset denoted by [152]. Through MAGMA we see that $Nt_1t_5t_1 = Nt_1t_3t_8 \in [138]$, $Nt_1t_5t_7 = Nt_1t_4t_2 \in [142]$, $Nt_1t_5t_4 = Nt_1t_6 \in [16]$ and $Nt_1t_5t_6 = Nt_1t_4 \in [14]$. Since 1 and 3 are in the same orbit of $N^{(15)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_5t_3 = Nt_1t_3t_8 \in [138]$. Similarly $Nt_1t_5t_{10} = Nt_1t_6 \in [16]$, $Nt_1t_5t_8 = Nt_1t_4 \in [14]$.

Now we must consider the coset stabilizer of $N^{(16)}$. $N^{(16)} = \{e, (2,10)(3,9)(4,8)(5,7)\}$. The number of single cosets in [16] can be found by $\frac{|N|}{|N^{(16)}|} = \frac{20}{2} = 10$. The orbits of $N^{(16)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}, \{6\}, \{2,10\}, \{3,9\}, \{4,8\}$ and $\{5,7\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $6 \in \{6\}$, $2 \in \{2,10\}$, $3 \in \{3,9\}$, $4 \in \{4,8\}$, and $5 \in \{5,7\}$. We note that $Nt_1t_6t_6 = Nt_1t_6^2 = Nt_1 \in [1]$ and $Nt_1t_6t_1$ belongs to a new double coset. Let [161] denote the new double coset to which $Nt_1t_6t_1$ belongs. Through MAGMA we see that $Nt_1t_6t_2 = Nt_1t_6 \in [16]$, $Nt_1t_6t_3 = Nt_1t_4t_9 \in [149]$, $Nt_1t_6t_4 = Nt_1t_3t_8 \in [138]$ and $Nt_1t_6t_5 = Nt_1t_5 \in [15]$. Since 2 and 10 are in the same orbit of $N^{(16)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_6t_{10} = Nt_1t_6 \in [16]$. Similarly $Nt_1t_6t_9 = Nt_1t_4t_9 \in [149]$, $Nt_1t_6t_8 = Nt_1t_3t_8 \in [138]$, and $Nt_1t_6t_7 = Nt_1t_5 \in [15]$.

Now we must consider the coset stabilizer of $N^{(131)}$. $N^{(131)} = \{e,\}$. The number of single cosets in [131] can be found by $\frac{|N|}{|N^{(131)}|} = \frac{20}{1} = 20$. The orbits of $N^{(131)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}$, $7 \in \{7\}$, $8 \in \{8\}$, $9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_3t_1t_1 = Nt_1t_3t_1^2 = Nt_1t_3 \in [13]$. We see that $Nt_1t_3t_1t_3$, $Nt_1t_3t_1t_4$, $Nt_1t_3t_1t_5$, $Nt_1t_3t_1t_6$, and $Nt_1t_3t_1t_8$ belong to new double cosets. Let [1313], [1314], [1315], [1316], and [1318] denote them respectively. Through MAGMA we find that $Nt_1t_3t_1t_2 = Nt_1t_3t_7 \in [137]$, $Nt_1t_3t_1t_7 = Nt_1t_4t_1 \in [141]$, $Nt_1t_3t_1t_3 = Nt_1t_4t_7 \in [147]$, and $Nt_1t_3t_1t_{10} = Nt_1t_4 \in [14]$.

Now we must consider the coset stabilizer of $N^{(137)}$. $N^{(137)} = \{e, (1,7)(2,6)(3,5)(8,10)\}$. The number of single cosets in [137] can be found by $\frac{|N|}{|N^{(137)}|} = \frac{20}{2} = 10$. The orbits of $N^{(137)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{4\}$, $\{9\}$, $\{1,7\}$, $\{2,6\}$, $\{3,5\}$, and $\{8,10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $4 \in \{4\}$, $9 \in \{9\}$, $7 \in \{1,7\}$, $2 \in \{2,6\}$,

$3 \in \{3,5\}$, and $8 \in \{8,10\}$. We note that $Nt_1t_3t_7t_7 = Nt_1t_3t_7^2 = Nt_1t_3 \in [13]$ and $Nt_1t_3t_7t_4$ is a new double coset denoted [1374]. Through MAGMA we find that $Nt_1t_3t_7t_9 = Nt_1t_3t_1t_3 \in [1313]$, $Nt_1t_3t_7t_2 = Nt_1t_3t_8 \in [138]$, $Nt_1t_3t_7t_3 = Nt_1t_3t_1t_5 \in [1315]$, $Nt_1t_3t_7t_8 = Nt_1t_3t_1 \in [131]$. Since 1 and 7 are in the same orbit of $N^{(137)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_3t_7t_1 = Nt_1t_3t_7 \in [137]$. Similarly, $Nt_1t_3t_7t_6 \in [138]$, $Nt_1t_3t_7t_5 \in [1315]$, and $Nt_1t_3t_7t_{10} \in [131]$.

Now we must consider the coset stabilizer of $N^{(138)}$. $N^{(138)} = \{e\}$. The number of single cosets in [12] can be found by $\frac{|N|}{|N^{(138)}|} = \frac{20}{1} = 20$. The orbits of $N^{(138)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\}$, $\{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}$, $7 \in \{7\}$, $8 \in \{8\}$, $9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_3t_8t_8 = Nt_1t_3t_8^2 = Nt_1t_3 \in [13]$and $nt_1t_3t_8t_6$ is a new double coset denoted [1386]. Through MAGMA we find that $Nt_1t_3t_8t_1 = Nt_1t_5t_2 \in [152]$, $Nt_1t_3t_8t_2 = Nt_1t_5 \in [15]$, $Nt_1t_3t_8t_3 = Nt_1t_6 \in [16]$, $Nt_1t_3t_8t_4 = Nt_1t_4t_9 \in [149]$, $Nt_1t_3t_8t_5 = Nt_1t_3t_1t_6 \in [1316]$, $Nt_1t_3t_8t_7 = Nt_1t_3t_8 \in [138]$, $Nt_1t_3t_8t_9 = Nt_1t_3t_7 \in [137]$, and $Nt_1t_3t_8t_{10} = Nt_1t_3t_1t_5 \in [1315]$.

Now we must consider the coset stabilizer of $N^{(141)}$. $N^{(141)} = \{e,\}$. The number of single cosets in [141] can be found by $\frac{|N|}{|N^{(141)}|} = \frac{20}{1} = 20$. The orbits of $N^{(141)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}$, $7 \in \{7\}$, $8 \in \{8\}$, $9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_4t_1t_1 = Nt_1t_4t_1^2 = Nt_1t_4 \in [14]$. We see that $Nt_1t_4t_1t_3$, $Nt_1t_4t_1t_4$, and $Nt_1t_4t_1t_5$ belong to new double cosets. Let [1413], [1414], and [1415] denote them respectively. Through MAGMA we find that $Nt_1t_4t_1t_2 = Nt_1t_4t_{10} \in [1410]$, $Nt_1t_4t_1t_6 = Nt_1t_3t_1t_6 \in [1316]$, $Nt_1t_4t_1t_7 = Nt_1t_3t_1 \in [131]$, $Nt_1t_4t_1t_8 = Nt_1t_3t_1t_8 \in [1318]$, $Nt_1t_4t_1t_9 = Nt_1t_3t_1t_4 \in [1314]$ and $Nt_1t_4t_1t_{10} = Nt_1t_4t_2 \in [142]$.

Now we must consider the coset stabilizer of $N^{(142)}$. $N^{(142)} = \{e,(2,10)(3,9)(4,8)(5,7)\}$. The number of single cosets in [142] can be found by $\frac{|N|}{|N^{(142)}|} = \frac{20}{2} = 10$. The orbits of $N^{(142)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\},\{6\},\{2,10\},\{3,9\},\{4,8\}$ and $\{5,7\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $6 \in \{6\}$, $2 \in \{2,10\}$, $3 \in \{3,9\}$, $4 \in \{4,8\}$, and

$5 \in \{5,7\}$. We note that $Nt_1t_4t_2t_2 = Nt_1t_4t_2^2 = Nt_1t_4 \in [14]$. Through MAGMA we see that $Nt_1t_4t_2t_1 = Nt_1t_5$, $Nt_1t_4t_2t_6 = Nt_1t_4t_2 \in [142]$, $Nt_1t_4t_2t_3 = Nt_1t_4t_1 \in [141]$, $Nt_1t_4t_2t_4 = Nt_1t_3t_1t_4 \in [1314]$ and $Nt_1t_4t_2t_5 = Nt_1t_3t_1t_8 \in [1318]$. Since 2 and 10 are in the same orbit of $N^{(142)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_4t_2t_{10} = Nt_1t_4 \in [14]$. Similarly $Nt_1t_4t_2t_9 = Nt_1t_4t_1 \in [141]$, $Nt_1t_4t_2t_8 = Nt_1t_3t_1t_4 \in [1314]$, and $Nt_1t_4t_2t_7 = Nt_1t_3t_8t_6 \in [1386]$.

Now we must consider the coset stabilizer of $N^{(147)}$. $N^{(147)} = \{e, (1,2)(3,10)(4,9)(5,8)(6,7)\}$. The number of single cosets in [147] can be found by $\frac{|N|}{|N^{(147)}|} = \frac{20}{2} = 10$. The orbits of $N^{(147)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1,2\}, \{3,10\}, \{4,9\}, \{5,8\}$ and $\{6,7\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1,2\}$, $3 \in \{3,10\}$, $4 \in \{4,9\}$, $5 \in \{5,8\}$, and $7 \in \{6,7\}$. We note that $Nt_1t_4t_7t_7 = Nt_1t_4t_7^2 = Nt_1t_4 \in [14]$. Through MAGMA we see that $Nt_1t_4t_7t_1 = Nt_1t_5t_2 \in [152]$, $Nt_1t_4t_7t_3 = Nt_1t_4t_1t_5 \in [1415]$, $Nt_1t_4t_7t_4 = Nt_1t_3t_1t_8 \in [1318]$, and $Nt_1t_4t_7t_5 = Nt_1t_3t_1 \in [131]$. Since 1 and 2 are in the same orbit of $N^{(147)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_4t_7t_2 = Nt_1t_5t_2 \in [152]$. Similarly $Nt_1t_4t_7t_{10} = Nt_1t_4t_1t_5 \in [1415]$, $Nt_1t_4t_7t_9 = Nt_1t_3t_1t_8 \in [1318]$, $Nt_1t_4t_7t_8 = Nt_1t_3t_1 \in [131]$, and $Nt_1t_4t_7t_6 = Nt_1t_4 \in [14]$.

Now we must consider the coset stabilizer of $N^{(149)}$. $N^{(149)} = \{e, (1,8)(2,7)(3,6)(4,5)(9,10)\}$. The number of single cosets in [149] can be found by $\frac{|N|}{|N^{(149)}|} = \frac{20}{2} = 10$. The orbits of $N^{(149)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1,8\}, \{2,7\}, \{3,6\}, \{4,5\}$ and $\{9,10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1,8\}$, $2 \in \{2,7\}$, $3 \in \{3,6\}$, $4 \in \{4,5\}$, and $9 \in \{9,10\}$. We note that $Nt_1t_4t_9t_9 = Nt_1t_4t_9^2 = Nt_1t_4 \in [14]$. Through MAGMA we see that $Nt_1t_4t_9t_1 = Nt_1t_4t_{10} \in [1410]$, $Nt_1t_4t_9t_2 = Nt_1t_3t_1t_6 \in [1316]$, $Nt_1t_4t_9t_3 = Nt_1t_3t_8 \in [138]$, and $Nt_1t_4t_9t_4 = Nt_1t_6 \in [16]$. Since 1 and 8 are in the same orbit of $N^{(149)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_4t_9t_8 = Nt_1t_4t_{10} \in [1410]$. Similarly $Nt_1t_4t_9t_7 = Nt_1t_3t_1t_6 \in [1316]$, $Nt_1t_4t_9t_6 = Nt_1t_3t_8 \in [138]$, $Nt_1t_4t_9t_5 = Nt_1t_6 \in [16]$ and $Nt_1t_4t_9t_{10} = Nt_1t_4 \in [14]$.

Now we must consider the coset stabilizer of $N^{(1410)}$. $N^{(1410)} = \{e, (1,5)(2,4)(6,10)(7,9)\}$. The number of single cosets in [1410] can be found by $\frac{|N|}{|N^{(1410)}|} = \frac{20}{2} = 10$. The orbits of $N^{(1410)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{3\}$, $\{8\}$, $\{1,5\}, \{2,4\}, \{6,10\}$, and $\{7,9\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $3 \in \{3\}$, $8 \in \{8\}$, $1 \in \{1,5\}$, $2 \in \{2,4\}$, $10 \in \{6,10\}$, and $7 \in \{7,9\}$. We note that $Nt_1t_4t_{10}t_{10} = Nt_1t_4t_{10}^2 = Nt_1t_4 \in [14]$ and $Nt_1t_4t_{10}t_3$ is a new double coset denoted [14103]. Through MAGMA we see that $Nt_1t_4t_{10}t_8 = Nt_1t_4t_1t_3 \in [1413]$, $Nt_1t_4t_{10}t_1 = Nt_1t_4t_9 \in [149]$, $Nt_1t_4t_{10}t_2 = Nt_1t_3t_1t_6 \in [1316]$, and $Nt_1t_4t_{10}t_7 = Nt_1t_4t_1 \in [141]$. Since 1 and 5 are in the same orbit of $N^{(1410)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_4t_{10}t_5 = Nt_1t_4t_9 \in [149]$. Similarly $Nt_1t_4t_{10}t_4 = Nt_1t_3t_1t_6 \in [1316]$, $Nt_1t_4t_{10}t_6 = Nt_1t_4 \in [14]$, and $Nt_1t_4t_{10}t_9 = Nt_1t_4t_1 \in [141]$.

Now we must consider the coset stabilizer of $N^{(152)}$. $N^{(152)} = \{e, (1,3)(4,10)(5,9)(6,8)\}$. The number of single cosets in [152] can be found by $\frac{|N|}{|N^{(152)}|} = \frac{20}{2} = 10$. The orbits of $N^{(152)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{2\}$, $\{7\}$, $\{1,3\}, \{4,10\}, \{5,9\}$, and $\{6,8\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $2 \in \{2\}$, $7 \in \{7\}$, $1 \in \{1,3\}$, $4 \in \{4,10\}$, $5 \in \{5,9\}$, and $6 \in \{6,8\}$. We note that $Nt_1t_5t_2t_2 = Nt_1t_5t_2^2 = Nt_1t_5 \in [15]$. Through MAGMA we see that $Nt_1t_5t_2t_7 = Nt_1t_5t_2 \in [152]$ $Nt_1t_5t_2t_1 = Nt_1t_3t_8 \in [138]$, $Nt_1t_5t_2t_4 = Nt_1t_3t_1t_5 \in [1315]$, $Nt_1t_5t_2t_5 = Nt_1t_4t_1t_5 \in [1415]$, and $Nt_1t_5t_2t_6 = Nt_1t_4t_7 \in [147]$. Since 1 and 3 are in the same orbit of $N^{(152)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_5t_2t_3 = Nt_1t_3t_8 \in [138]$. Similarly $Nt_1t_5t_2t_{10} = Nt_1t_3t_1t_5 \in [1315]$, $Nt_1t_5t_2t_9 = Nt_1t_4t_1t_5 \in [1415]$, and $Nt_1t_5t_2t_8 = Nt_1t_4t_7 \in [147]$.

Now we must consider the coset stabilizer of $N^{(152)}$. $N^{(152)} = \{e, (1,3)(4,10)(5,9)(6,8)\}$. The number of single cosets in [152] can be found by $\frac{|N|}{|N^{(152)}|} = \frac{20}{2} = 10$. The orbits of $N^{(152)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{2\}$, $\{7\}$, $\{1,3\}, \{4,10\}, \{5,9\}$, and $\{6,8\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $2 \in \{2\}$, $7 \in \{7\}$, $1 \in \{1,3\}$, $4 \in \{4,10\}$, $5 \in \{5,9\}$, and $6 \in \{6,8\}$. We note that $Nt_1t_5t_2t_2 = Nt_1t_5t_2^2 = Nt_1t_5 \in [15]$. Through MAGMA we see that $Nt_1t_5t_2t_7 = Nt_1t_5t_2 \in [152]$ $Nt_1t_5t_2t_1 = Nt_1t_3t_8 \in [138]$, $Nt_1t_5t_2t_4 = Nt_1t_3t_1t_5 \in [1315]$, $Nt_1t_5t_2t_5 = Nt_1t_4t_1t_5 \in [1415]$, and $Nt_1t_5t_2t_6 = Nt_1t_4t_7 \in [147]$. Since 1 and 3 are in the same orbit of $N^{(152)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_5t_2t_3 = Nt_1t_3t_8 \in [138]$. Similarly $Nt_1t_5t_2t_{10} = Nt_1t_3t_1t_5 \in [1315]$, $Nt_1t_5t_2t_9 = Nt_1t_4t_1t_5 \in [1415]$, and $Nt_1t_5t_2t_8 = Nt_1t_4t_7 \in [147]$.

Now we must consider the coset stabilizer of $N^{(161)}$. $N^{(161)} = \{e$, (1, 3)(4, 10)(5, 9)(6, 8), (1, 3, 5, 7, 9)(2, 4, 6, 8, 10), (1, 8, 5, 2, 9, 6, 3, 10, 7, 4), (1, 8)(2, 7)(3, 6)(4, 5)(9, 10), (1, 6)(2, 7)(3, 8)(4, 9)(5, 10), (1, 6)(2, 5)(3, 4)(7, 10)(8, 9), (1, 4, 7, 10, 3, 6, 9, 2, 5, 8), (1, 4)(2, 3)(5, 10)(6, 9)(7, 8), (1, 9, 7, 5, 3)(2, 10, 8, 6, 4), (1, 9)(2, 8)(3, 7)(4, 6), (1, 7, 3, 9, 5)(2, 8, 4, 10, 6), (1, 7)(2, 6)(3, 5)(8, 10), (1, 2, 3, 4, 5, 6, 7, 8, 9, 10), (1, 2)(3, 10)(4, 9)(5, 8)(6, 7), (1, 10, 9, 8, 7, 6, 5, 4, 3, 2), (1, 10)(2, 9)(3, 8)(4, 7)(5, 6), (1, 5, 9, 3, 7)(2, 6, 10, 4, 8) (1, 5)(2, 4)(6, 10)(7, 9), (2, 10)(3, 9)(4, 8)(5, 7)}. The number of single cosets in [161] can be found by $\frac{|N|}{|N^{(161)}|} = \frac{20}{20} = 1$. The orbit of $N^{(161)}$ is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Now we select a representative the orbit and determine the double coset it belongs to. Let us select $1 \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. We notice that $Nt_1t_6t_1t_1 = Nt_1t_6t_1^2 = Nt_1t_6 \in [16]$.

Now we must consider the coset stabilizer of $N^{(1313)}$. $N^{(1313)} = \{e, (1, 7)(2, 6)(3, 5)(8, 10)\}$. The number of single cosets in [152] can be found by $\frac{|N|}{|N^{(1313)}|} = \frac{20}{2} = 10$. The orbits of $N^{(1313)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are $\{4\}$, $\{9\}$, $\{1, 7\}, \{2, 6\}, \{3, 5\}$, and $\{8, 10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $4 \in \{4\}$, $9 \in \{9\}$, $1 \in \{1, 7\}$, $2 \in \{2, 6\}$, $3 \in \{3, 5\}$, and $8 \in \{8, 10\}$. We note that $Nt_1t_3t_1t_3t_3 = Nt_1t_3t_1t_3^2 = Nt_1t_3t_1 \in [131]$ and $Nt_1t_3t_1t_3t_8$ is a new double coset which we will denote [13138]. Through MAGMA we see that $Nt_1t_3t_1t_3t_4 = Nt_1t_3t_7 \in [137]$ $Nt_1t_3t_1t_3t_9 = Nt_1t_4t_1t_3 \in [1413]$, $Nt_1t_3t_1t_3t_1 = Nt_1t_3t_1t_8 \in [1318]$, and $Nt_1t_3t_1t_3t_2 = Nt_1t_3t_1t_4 \in [1314]$. Since 1 and 7 are in the same orbit of $N^{(1313)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $Nt_1t_3t_1t_3t_7 = Nt_1t_3t_1t_8 \in [1318]$. Similarly $Nt_1t_3t_1t_3t_6 = Nt_1t_3t_1t_4 \in [1314]$, $Nt_1t_3t_1t_3t_5 = Nt_1t_3t_1 \in [131]$, and $Nt_1t_3t_1t_3t_{10} = Nt_1t_3t_1t_3t_8 \in [13138]$.

Now we must consider the coset stabilizer of $N^{(1314)}$. $N^{(1314)} = \{e, \}$. The number of single cosets in [1314] can be found by $\frac{|N|}{|N^{(1314)}|} = \frac{20}{1} = 20$. The orbits of $N^{(1314)}$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}$, $7 \in \{7\}$, $8 \in \{8\}$, $9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_3t_1t_4t_4 = Nt_1t_3t_1t_4^2 = Nt_1t_3t_1 \in [131]$. We see that $Nt_1t_3t_1t_4t_1$ and $Nt_1t_3t_1t_4t_2$ belong to new double cosets. Let [13141] and [13142] denote them respectively. Through MAGMA we find that $Nt_1t_3t_1t_4t_3 = Nt_1t_3t_1t_5 \in [1315]$, $Nt_1t_3t_1t_4t_5 = Nt_1t_3t_1t_3 \in [1313]$, $Nt_1t_3t_1t_4t_6 = Nt_1t_3t_1t_8 \in [1318]$, $Nt_1t_3t_1t_4t_7 = Nt_1t_4t_1 \in [141]$, $t_1t_3t_1t_4t_8 = Nt_1t_4t_2 \in [142]$, $Nt_1t_3t_1t_4t_9 = Nt_1t_3t_8t_6 \in [1386]$ and $Nt_1t_3t_1t_4t_{10} = Nt_1t_3t_1t_3t_8 \in [13138]$.

Now we must consider the coset stabilizer of $N^{(1315)}$. $N^{(1315)} = \{e, \}$. The number of single cosets in [1315] can be found by $\frac{|N|}{|N^{(1315)}|} = \frac{20}{1} = 20$. The orbits of $N^{(1315)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}$, $7 \in \{7\}$, $8 \in \{8\}$, $9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_3t_1t_5t_5 = Nt_1t_3t_1t_5^2 = Nt_1t_3t_1 \in [131]$. Through MAGMA we find that $Nt_1t_3t_1t_5t_1 = Nt_1t_5t_2 \in [152]$, $Nt_1t_3t_1t_5t_2 = Nt_1t_4t_1t_5 \in [1415]$, $Nt_1t_3t_1t_5t_3 = Nt_1t_3t_1t_3t_8 \in [13138]$, $Nt_1t_3t_1t_5t_4 = Nt_1t_3t_1t_6 \in [1316]$, $Nt_1t_3t_1t_5t_6 = Nt_1t_3t_1t_4 \in [1314]$, $Nt_1t_3t_1t_5t_7 = Nt_1t_3t_1t_4t_2 \in [13142]$, $t_1t_3t_1t_5t_8 = Nt_1t_3t_7t_4 \in [1374]$, $Nt_1t_3t_1t_5t_9 = Nt_1t_3t_7 \in [137]$ and $Nt_1t_3t_1t_5t_{10} = Nt_1t_3t_8 \in [138]$.

Now we must consider the coset stabilizer of $N^{(1316)}$. $N^{(1316)} = \{e,\}$. The number of single cosets in [1316] can be found by $\frac{|N|}{|N^{(1316)}|} = \frac{20}{1} = 20$. The orbits of $N^{(1316)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}$, $7 \in \{7\}$, $8 \in \{8\}$, $9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_3t_1t_6t_6 = Nt_1t_3t_1t_6^2 = Nt_1t_3t_1 \in [131]$. Through MAGMA we find that $Nt_1t_3t_1t_6t_1 = Nt_1t_4t_9 \in [149]$, $Nt_1t_3t_1t_6t_2 = Nt_1t_4t_{10} \in [1410]$, $Nt_1t_3t_1t_6t_3 = Nt_1t_4t_{10}t_3 \in [14103]$, $Nt_1t_3t_1t_6t_4 = Nt_1t_4t_1t_5 \in [1415]$, $Nt_1t_3t_1t_6t_5 = Nt_1t_4t_1 \in [141]$, $Nt_1t_3t_1t_6t_7 = Nt_1t_3t_1t_5 \in [1315]$, $t_1t_3t_1t_6t_8 = Nt_1t_3t_1t_3t_8 \in [13138]$, $Nt_1t_3t_1t_6t_9 = Nt_1t_3t_8t_6 \in [1386]$ and $Nt_1t_3t_1t_6t_{10} = Nt_1t_3t_8 \in [138]$.

Now we must consider the coset stabilizer of $N^{(1318)}$. $N^{(1318)} = \{e,\}$. The number of single cosets in [1318] can be found by $\frac{|N|}{|N^{(1318)}|} = \frac{20}{1} = 20$. The orbits of $N^{(1318)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}$, $7 \in \{7\}$, $8 \in \{8\}$, $9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_3t_1t_8t_8 = Nt_1t_3t_1t_8^2 = Nt_1t_3t_1 \in [131]$ and $Nt_1t_3t_1t_8t_5$ is a new double coset denoted [13185]. Through MAGMA we find that $Nt_1t_3t_1t_8t_1 = Nt_1t_3t_1t_3 \in [1313]$, $Nt_1t_3t_1t_8t_2 = Nt_1t_3t_1t_3t_8 \in [13138]$, $Nt_1t_3t_1t_8t_3 = Nt_1t_4t_{10}t_3 \in [14103]$, $Nt_1t_3t_1t_8t_4 = Nt_1t_3t_7t_4 \in [1374]$, $Nt_1t_3t_1t_8t_6 = Nt_1t_4t_1t_5 \in [1415]$, $t_1t_3t_1t_8t_7 = Nt_1t_4t_7 \in [147]$, $Nt_1t_3t_1t_8t_9 = Nt_1t_4t_1 \in [141]$ and $Nt_1t_3t_1t_8t_{10} = Nt_1t_3t_1t_4 \in [1314]$.

Now we must consider the coset stabilizer of $N^{(1374)}$. $N^{(1374)} = \{e, (1,7)(2,6)(3,5)(8,10)\}$. The number of single cosets in [1374] can be found by $\frac{|N|}{|N^{(1374)}|} = \frac{20}{2} = 10$. The orbits of $N^{(1374)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{4\}$, $\{9\}$, $\{1,7\}$, $\{2,6\}$, $\{3,5\}$, and $\{8,10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $4 \in \{4\}$, $9 \in \{9\}$, $7 \in \{1,7\}$, $2 \in \{2,6\}$, $3 \in \{3,5\}$, and $8 \in \{8,10\}$. We note that $Nt_1t_3t_7t_4t_4 = Nt_1t_3t_7t_4^2 = Nt_1t_3 \in [13]$. Through MAGMA we find that $Nt_1t_3t_7t_4t_9 = Nt_1t_4t_{10}t_3 \in [14103]$, $Nt_1t_3t_7t_4t_1 = Nt_1t_3t_1t_8t_5 \in [13185]$, $Nt_1t_3t_7t_4t_2 = Nt_1t_3t_1t_4t_2 \in [13142]$, $Nt_1t_3t_7t_4t_3 = Nt_1t_3t_1t_5 \in [1315]$, $Nt_1t_3t_7t_4t_8 = Nt_1t_3t_1t_8 \in [1318]$. Since 1 and 7 are in the same orbit of $N^{(1374)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_3t_7t_4t_7 = Nt_1t_3t_1t_8t_5 \in [13185]$. Similarly, $Nt_1t_3t_7t_4t_6 = Nt_1t_3t_1t_4t_2 \in [13142]$, $Nt_1t_3t_7t_4t_5 = Nt_1t_3t_1t_5 \in [1315]$, and $Nt_1t_3t_7t_4t_{10} = Nt_1t_3t_1t_8 \in [1318]$.

Now we must consider the coset stabilizer of $N^{(1386)}$. $N^{(1386)} = \{e, (1,10)(2,9)(3,8)(4,7)(5,6)\}$. The number of single cosets in [1386] can be found by $\frac{|N|}{|N^{(1386)}|} = \frac{20}{2} = 10$. The orbits of $N^{(1386)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1,10\}, \{2,9\}, \{3,8\}, \{4,7\}$ and $\{5,6\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1,10\}$, $2 \in \{2,9\}$, $3 \in \{3,8\}$, $4 \in \{4,7\}$, and $6 \in \{5,6\}$. We note that $Nt_1t_3t_8t_6t_6 = Nt_1t_3t_8t_6^2 = Nt_1t_3t_8 \in [138]$. Through MAGMA we see that $Nt_1t_3t_8t_6t_1 = Nt_1t_4t_2 \in [142]$, $Nt_1t_3t_8t_6t_2 = Nt_1t_3t_1t_4 \in [1314]$, $Nt_1t_3t_8t_6t_3 = Nt_1t_3t_1t_3t_8 \in [13138]$, and $Nt_1t_3t_8t_6t_4 = Nt_1t_3t_1t_6 \in [1316]$. Since 1 and 10 are in the same orbit of $N^{(1386)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_3t_8t_6t_{10} = Nt_1t_4t_2 \in [142]$. Similarly $Nt_1t_3t_8t_6t_9 = Nt_1t_3t_1t_4 \in [1314]$, $Nt_1t_3t_8t_6t_8 = Nt_1t_3t_1t_3t_8 \in [13138]$, $Nt_1t_3t_8t_6t_7 = Nt_1t_3t_1t_6 \in [1316]$ and $Nt_1t_3t_8t_6t_5 = Nt_1t_3t_8 \in [138]$.

Now we must consider the coset stabilizer of $N^{(1413)}$. $N^{(1413)} = \{e, (1,7)(2,6)(3,5)(8,10)\}$. The number of single cosets in [1413] can be found by $\frac{|N|}{|N^{(1413)}|} = \frac{20}{2} = 10$. The orbits of $N^{(1413)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{4\}$, $\{9\}$, $\{1,7\}$, $\{2,6\}$, $\{3,5\}$, and $\{8,10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $4 \in \{4\}$, $9 \in \{9\}$, $7 \in \{1,7\}$, $2 \in \{2,6\}$, $3 \in \{3,5\}$, and $8 \in \{8,10\}$. We note that $Nt_1t_4t_1t_3t_3 = Nt_1t_4t_1t_3^2 = Nt_1t_4t_1 \in [141]$. Through MAGMA we find that $Nt_1t_4t_1t_3t_4 = Nt_1t_4t_{10} \in [1410]$, $Nt_1t_4t_1t_3t_9 = Nt_1t_3t_1t_3 \in [1313]$, $Nt_1t_4t_1t_3t_1 = Nt_1t_3t_1t_4t_1 \in [13141]$, $Nt_1t_4t_1t_3t_2 = Nt_1t_4t_1t_4 \in [1414]$, $Nt_1t_4t_1t_3t_8 = Nt_1t_3t_1t_3t_8 \in [13138]$. Since 1 and 7 are in the same orbit of

$N^{(1413)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_4t_1t_3t_7 = Nt_1t_3t_1t_4t_1 \in [13141]$. Similarly, $Nt_1t_4t_1t_3t_6 = Nt_1t_4t_1t_4 \in [1414]$, $Nt_1t_4t_1t_3t_5 = Nt_1t_4t_1 \in [141]$, and $Nt_1t_4t_1t_3t_{10} = Nt_1t_3t_1t_3t_8 \in [13138]$.

Now we must consider the coset stabilizer of $N^{(1414)}$. $N^{(1414)} = \{e, (2,10)(3,9)(4,8)(5,7)\}$. The number of single cosets in [1414] can be found by $\frac{|N|}{|N^{(1414)}|} = \frac{20}{2} = 10$. The orbits of $N^{(1414)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\},\{6\},\{2,10\},\{3,9\},\{4,8\}$ and $\{5,7\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $6 \in \{6\}$, $2 \in \{2,10\}$, $3 \in \{3,9\}$, $4 \in \{4,8\}$, and $5 \in \{5,7\}$. We note that $Nt_1t_4t_1t_4t_4 = Nt_1t_4t_1t_4^2 = Nt_1t_4t_1 \in [141]$ and $Nt_1t_4t_1t_4t_1$ is a new double coset denoted [14141]. Through MAGMA we see that $Nt_1t_4t_1t_4t_6 = Nt_1t_3t_1t_4t_1 \in [13141]$, $Nt_1t_4t_1t_4t_2 = Nt_1t_3t_1t_8t_5 \in [13185]$, $Nt_1t_4t_1t_4t_3 = Nt_1t_4t_1t_5 \in [1415]$ and $Nt_1t_4t_1t_4t_5 = Nt_1t_4t_1t_3 \in [1413]$. Since 2 and 10 are in the same orbit of $N^{(1414)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_4t_1t_4t_{10} = Nt_1t_3t_1t_8t_5 \in [13185]$. Similarly $Nt_1t_4t_1t_4t_9 = Nt_1t_4t_1t_5 \in [1415]$, $Nt_1t_4t_1t_4t_8 = Nt_1t_4t_1 \in [141]$, and $Nt_1t_4t_1t_4t_7 = Nt_1t_4t_1t_3 \in [1413]$.

Now we must consider the coset stabilizer of $N^{(1415)}$. $N^{(1415)} = \{e,\}$. The number of single cosets in [1415] can be found by $\frac{|N|}{|N^{(1415)}|} = \frac{20}{1} = 20$. The orbits of $N^{(1415)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}$, $7 \in \{7\}$, $8 \in \{8\}$, $9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_4t_1t_5t_5 = Nt_1t_4t_1t_5^2 = Nt_1t_4t_1 \in [141]$. Through MAGMA we find that $Nt_1t_4t_1t_5t_1 = Nt_1t_3t_1t_5 \in [1315]$, $Nt_1t_4t_1t_5t_2 = Nt_1t_3t_1t_3t_8 \in [13138]$, $Nt_1t_4t_1t_5t_3 = Nt_1t_4t_{10}t_3 \in [14103]$, $Nt_1t_4t_1t_5t_4 = Nt_1t_3t_1t_6 \in [1316]$, $Nt_1t_4t_1t_5t_6 = Nt_1t_4t_1t_4 \in [1414]$, $Nt_1t_4t_1t_5t_7 = Nt_1t_3t_1t_8t_5 \in [13185]$, $t_1t_4t_1t_5t_8 = Nt_1t_3t_1t_8 \in [1318]$, $Nt_1t_4t_1t_5t_9 = Nt_1t_4t_7 \in [147]$ and $Nt_1t_4t_1t_5t_{10} = Nt_1t_5t_2 \in [152]$.

Now we must consider the coset stabilizer of $N^{(14103)}$. $N^{(14103)} = \{e, (1,5)(2,4)(6,10)(7,9)\}$. The number of single cosets in [14103] can be found by $\frac{|N|}{|N^{(14103)}|} = \frac{20}{2} = 10$. The orbits of $N^{(14103)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{3\},\{8\},\{1,5\},\{2,4\},\{6,10\}$, and $\{7,9\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $3 \in \{3\}$, $8 \in \{8\}$, $1 \in \{1,5\}$, $2 \in \{2,4\}$, $10 \in \{6,10\}$, and $7 \in \{7,9\}$. We note that $Nt_1t_4t_{10}t_3t_3 = Nt_1t_4t_{10}t_3^2 = Nt_1t_4t_{10} \in [1410]$. Through

MAGMA we see that $Nt_1t_4t_{10}t_8 = Nt_1t_3t_7t_4 \in [1374]$, $Nt_1t_4t_{10}t_1 = Nt_1t_4t_1t_5 \in [1415]$, $Nt_1t_4t_{10}t_3t_22 = Nt_1t_3t_1t_6 \in [1316]$, $Nt_1t_4t_{10}t_3t_6 = Nt_1t_3t_1t_3 \in [1313]$ and $Nt_1t_4t_{10}t_3t_7 = Nt_1t_3t_1t_8 \in [1318]$. Since 1 and 5 are in the same orbit of $N^{(14103)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_4t_{10}t_5 = Nt_1t_4t_1t_5 \in [1415]$. Similarly $Nt_1t_4t_{10}t_3t_4 = Nt_1t_3t_1t_6 \in [1316]$, $Nt_1t_4t_{10}t_3t_{10} = Nt_1t_3t_1t_3 \in [1313]$, and $Nt_1t_4t_{10}t_3t_9 = Nt_1t_3t_1t_8 \in [1318]$.

Now we must consider the coset stabilizer of $N^{(13138)}$. $N^{(13138)} = \{e,\}$. The number of single cosets in [13138] can be found by $\frac{|N|}{|N^{(13138)}|} = \frac{20}{1} = 20$. The orbits of $N^{(13138)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, and $\{10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$, $5 \in \{5\}$, $6 \in \{6\}$, $7 \in \{7\}$, $8 \in \{8\}$, $9 \in \{9\}$, and $10 \in \{10\}$. We note that $Nt_1t_3t_1t_3t_8t_8 = Nt_1t_3t_1t_3t_8^2 = Nt_1t_3t_1t_3 \in [1313]$. Through MAGMA we find that $Nt_1t_3t_1t_3t_8t_1 = Nt_1t_4t_1t_5 \in [1415]$, $Nt_1t_3t_1t_3t_8t_2 = Nt_1t_3t_1t_5 \in [1315]$, $Nt_1t_3t_1t_3t_8t_3 = Nt_1t_3t_1t_6 \in [1316]$, $Nt_1t_3t_1t_3t_8t_4 = Nt_1t_3t_8t_6 \in [1386]$, $t_1t_3t_1t_3t_8t_5 = Nt_1t_3t_1t_4 \in [1314]$, $Nt_1t_3t_1t_3t_8t_6 = Nt_1t_3t_1t_4t_1 \in [13141]$, $Nt_1t_3t_1t_3t_8t_7 = Nt_1t_4t_1t_3 \in [1413]$, $Nt_1t_3t_1t_3t_8t_9 = Nt_1t_3t_1t_8 \in [1318]$ and $Nt_1t_3t_1t_3t_8t_{10} = Nt_1t_4t_{10}t_3 \in [14103]$.

Now we must consider the coset stabilizer of $N^{(13141)}$. $N^{(13141)} = \{e, (1,7)(2,6)(3,5)(8,10)\}$. The number of single cosets in [13141] can be found by $\frac{|N|}{|N^{(13141)}|} = \frac{20}{2} = 10$. The orbits of $N^{(13141)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{4\}$, $\{9\}$, $\{1,7\}$, $\{2,6\}$, $\{3,5\}$, and $\{8,10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $4 \in \{4\}$, $9 \in \{9\}$, $7 \in \{1,7\}$, $2 \in \{2,6\}$, $3 \in \{3,5\}$, and $8 \in \{8,10\}$. We note that $Nt_1t_3t_1t_4t_1t_1 = Nt_1t_3t_1t_4t_1^2 = Nt_1t_3t_1t_4 \in [1314]$ and $Nt_1t_3t_1t_4t_1t_9$ is a new double coset denoted [131419]. Through MAGMA we find that $Nt_1t_3t_1t_4t_1t_4 = Nt_1t_4t_1t_4 \in [1414]$, $Nt_1t_3t_1t_4t_1t_2 = Nt_1t_3t_1t_3t_8 \in [13138]$, $Nt_1t_3t_1t_3t_8t_3 = Nt_1t_4t_1t_3 \in [1413]$, $Nt_1t_4t_1t_3t_2 = Nt_1t_4t_1t_4 \in [1414]$, and $Nt_1t_4t_1t_3t_8 = Nt_1t_3t_1t_3t_8 \in [13138]$. Since 1 and 7 are in the same orbit of $N^{(1413)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_4t_1t_3t_7 = Nt_1t_3t_1t_4t_1 \in [13141]$. Similarly, $Nt_1t_4t_1t_3t_6 = Nt_1t_4t_1t_4 \in [1414]$, $Nt_1t_4t_1t_3t_5 = Nt_1t_4t_1 \in [141]$, and $Nt_1t_4t_1t_3t_{10} = Nt_1t_3t_1t_3t_8 \in [13138]$.

Now we must consider the coset stabilizer of $N^{(13142)}$. $N^{(13142)} = \{e, (1,7)(2,6)(3,5)(8,10)\}$. The number of single cosets in [13142] can be found by $\frac{|N|}{|N^{(13142)}|} = \frac{20}{2} = 10$.

The orbits of $N^{(13142)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ are $\{4\}$, $\{9\}$, $\{1,7\}$, $\{2,6\}$, $\{3,5\}$, and $\{8,10\}$. Now we select a representative t_i from each orbit and determine the double coset containing each t_i. Let us select $4 \in \{4\}$, $9 \in \{9\}$, $7 \in \{1,7\}$, $2 \in \{2,6\}$, $3 \in \{3,5\}$, and $8 \in \{8,10\}$. We note that $Nt_1t_3t_1t_4t_2t_2 = Nt_1t_3t_1t_4t_2^2 = Nt_1t_3t_1t_4 \in [1314]$. Through MAGMA we find that $Nt_1t_3t_1t_4t_2t_4 = Nt_1t_3t_1t_4t_1t_9 \in [131419]$, $Nt_1t_3t_1t_4t_2t_9 = Nt_1t_3t_1t_8t_5 \in [13185]$, $Nt_1t_3t_1t_4t_2t_1 = Nt_1t_3t_1t_5 \in [1315]$, $Nt_1t_3t_1t_4t_2t_3 = Nt_1t_3t_1t_4t_1 \in [13141]$, and $Nt_1t_3t_1t_4t_2t_8 = Nt_1t_3t_7t_4 \in [1374]$. Since 1 and 7 are in the same orbit of $N^{(13142)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$, $Nt_1t_3t_1t_4t_2t_7 = Nt_1t_3t_1t_5 \in [1315]$. Similarly, $Nt_1t_3t_1t_4t_2t_6 = Nt_1t_4t_1t_4 \in [1414]$, $Nt_1t_3t_1t_4t_2t_5 = Nt_1t_3t_1t_4t_1 \in [13141]$, and $Nt_1t_3t_1t_4t_2t_{10} = Nt_1t_3t_7t_4 \in [1374]$.

Now we must consider the coset stabilizer of $N^{(14141)}$. $N^{(14141)} = \{e$, (1, 5)(2, 4)(6, 10)(7, 9), (1, 9, 7, 5, 3)(2, 10, 8, 6, 4), (1, 3)(4, 10)(5, 9)(6, 8), (1, 7)(2, 6)(3, 5)(8, 10), (1, 7, 3, 9, 5)(2, 8, 4, 10, 6), (1, 3, 5, 7, 9)(2, 4, 6, 8, 10), (2, 10)(3, 9)(4, 8)(5, 7), (1, 9)(2, 8)(3, 7)(4, 6), (1, 5, 9, 3, 7)(2, 6, 10, 4, 8)$\}$. The number of single cosets in [14141] can be found by $\frac{|N|}{|N^{(14141)}|} = \frac{20}{10} = 2$. The orbit of $N^{(14141)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ is $\{1,2,3,4,5,6,7,8,9,10\}$. Now we select a representative, say t_1 from the orbit and determine the double coset that contains it. We see that $Nt_1t_4t_1t_4t_1t_1 = Nt_1t_4t_1t_4t_1^2 = Nt_1t_4t_1t_4 \in [1414]$.

Now we must consider the coset stabilizer of $N^{(131419)}$. $N^{(131419)} = \{e$, (2, 10)(3, 9)(4, 8)(5, 7), (1, 3, 5, 7, 9)(2, 4, 6, 8, 10), (1, 9)(2, 8)(3, 7)(4, 6), (1, 3)(4, 10)(5, 9)(6, 8), (1, 5, 9, 3, 7)(2, 6, 10, 4, 8), (1, 9, 7, 5, 3)(2, 10, 8, 6, 4), (1, 7)(2, 6)(3, 5)(8, 10), (1, 5)(2, 4)(6, 10)(7, 9), (1, 7, 3, 9, 5)(2, 8, 4, 10, 6)$\}$. The number of single cosets in [131419] can be found by $\frac{|N|}{|N^{(131419)}|} = \frac{20}{10} = 2$. The orbit of $N^{(14141)}$ on $\{1,2,3,4,5,6,7,8,9,10\}$ is $\{1,2,3,4,5,6,7,8,9,10\}$. Now we select a representative, say t_9 from the orbit and determine the double coset that contains it. We see that $Nt_1t_3t_1t_4t_1t_9t_9 = Nt_1t_3t_1t_4t_1t_9^2 = Nt_1t_3t_1t_4t_1 \in [13141]$. We have no new double cosets and we see that we have the following Cayley Diagram.

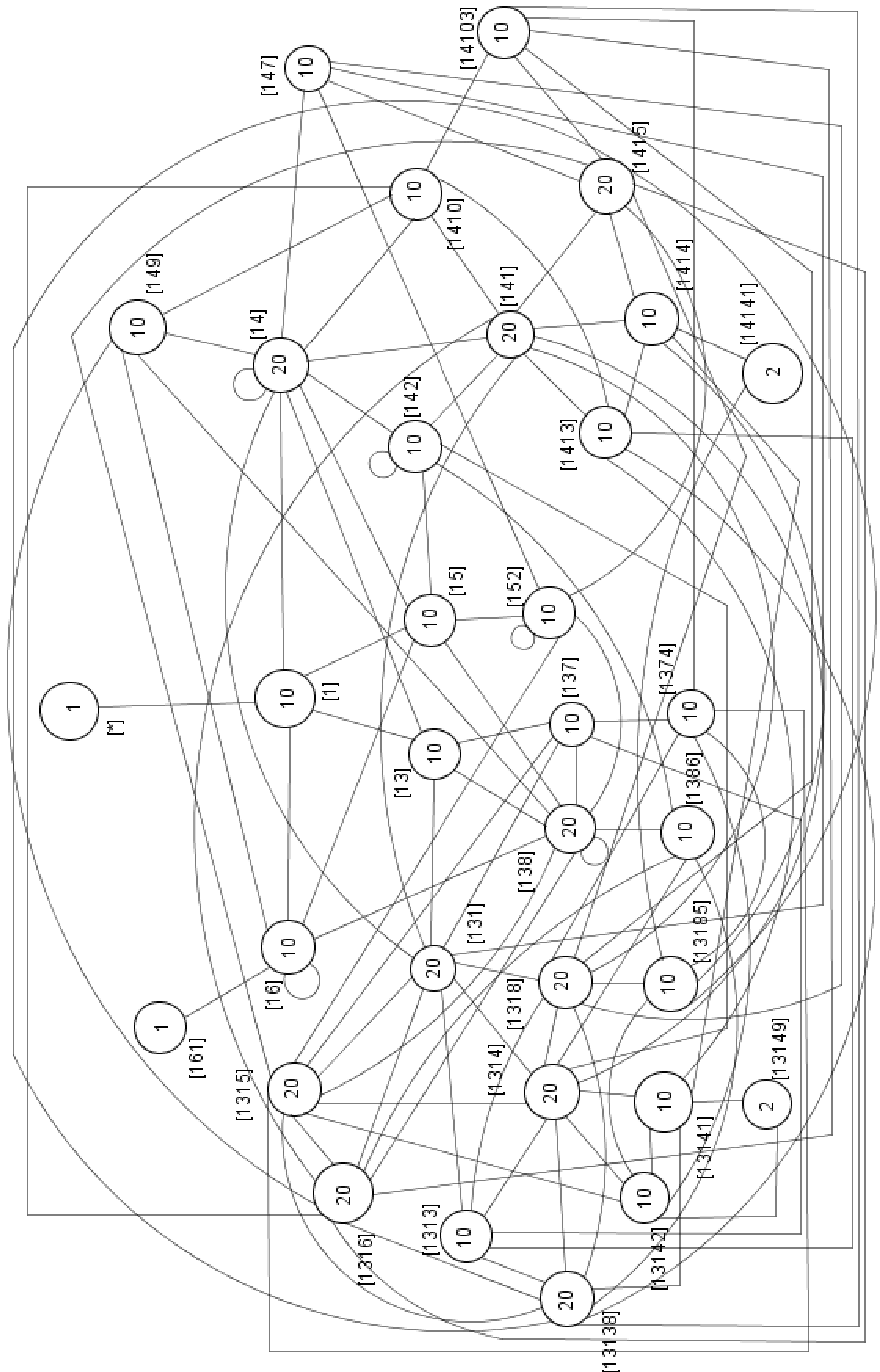

Figure 2.5: Cayley Diagram of $2 \times [3 \times L_2(3) : 2]$ over D_{10}

Chapter 3

Composition Factors

3.1 Preliminaries

In this chapter we will look at the isomorphic types of our finite groups. Every composition series gives us a product of simple groups that make up a group G. We want to write G as a product of finite simple subgroups that provide us with the isomorphic type of our progenitors when factored by specified relations.

Theorem 3.1. *G is an extension of K by Q if G has a normal subgroup $K_1 \cong K$ such that $G/K_1 \cong Q$, where $G = KQ$.*

Definition 3.2 (Center)**.** *The center of a group G, denoted by Z(G), is the set of all $a \in G$ that commute with every element of G.*

Definition 3.3 (Normal Series)**.** *A chain of subgroups of G, $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq ... \supseteq G_n = 1$ such that $G_i \trianglelefteq G$, $\forall$ i, $1 \leq i \leq n$ is called a normal series of G.*

Definition 3.4 (Subnormal Series)**.** *A chain of subgroups of G, $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq ... \supseteq G_n = 1$ such that $G_{i+1} \trianglelefteq G_i$, $\forall i$, $0 \leq i \leq n-1$ is called a subnormal series of G.*

Note 3.5. Any two composition series of a group are isomorphic.

Note 3.6. The composition factors $G_0/g_1, G_1/G_2, ..., G_{n-1}/G_n$ of the composition series $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq ... \supseteq G_{n-1} \supset G_n$ are simple.

List of Possible Extensions

1. Direct Product: G is a direct product of H by K if H and K are both normal in G and $G = H \times K$, where $H \cap K = 1$.

2. Semi-Direct Product: G is a semi-direct product of H by K if H is normal in G and $H \cap K = 1$. The semi-direct product is denoted $H : K$.

3. Central Extension: G is a central extension of H by K, denoted $H^{\bullet}K$if the following are true.

   (a) G is perfect

   (b) H is normal in G

   (c) H is the center of G

4. Mixed Extension: A mixed extension, denoted $G = H \bullet N$, is when you have an abelian group which is not the center of G.

3.2 Extension Problem $U_3(3) : 2$

Consider the group

$G =< a, b, t|a^2, b^3, (a*b)^7, (b*a*b^{-1}*a)^4, t^2, (t, b), (t, a*b*a*b^{-1}*a), (a*b^{-1}*a*t)^n, (a*b^{-1}*a*t^a)^o, (a*b^{-1}*a*t^{a^{b^2}})^p, (a*b^{-1}*a*b*t)^q, (a*b^-1*a*b*t^{a^{b^2}})^r >$

Using MAGMA we have the following composition factors

```
CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  2A(2, 3)                    = U(3, 3)
    1
```

So we have the composition series $G_1 \supseteq G_2 \supseteq 1$ where $G = (G_1/G_2) \cdot (G_2/1)$
$G = C_2 \cdot U_3(3)$.

The group also has the following normal lattice.

```
NL:=NormalLattice(G1);
NL;

Normal subgroup lattice
-----------------------

[3]  Order 12096  Length 1  Maximal Subgroups: 2
---
[2]  Order 6048   Length 1  Maximal Subgroups: 1
---
[1]  Order 1      Length 1  Maximal Subgroups:
```

First we check the normal lattice for a central extension and normally we also check to see if there is a direct product between the normal subgroups but we can see from the normal lattice that a direct product is not possible.

```
> Center(G1);
Permutation group acting on a set of cardinality 72
Order = 1
```

Now we check as to whether or not there is a semidirect product between Cyclic(2) and $U_3(3)$. We check if this is possible by running a loop to find all subgroups of G1 that are of order 2.

```
> S:=Subgroups(G1);
> for i in [1..#S] do if #S[i]`subgroup eq 2 then i;
for | if>end if; end for;
2
3
```

So we see that there does exist a subgroup of order 2. Now we look for an element of order 2 in G1 and outside NL[2].

```
>for r in NL[3] do if Order(r) eq 2 and r notin NL[2] and NL[3]
for | if>eq sub<NL[3]|NL[2],r>
for | if>then R:=r; break; end if; end for;

>R:=G1!(1, 6)(2, 15)(3, 32)(4, 8)(5, 61)(7, 28)(9, 42)(10, 44)
    (11, 50)(12, 17)(13, 46)(14, 20)(16, 69)(18, 36)(19, 38)(21,
    45)(22, 35)(23, 26)(24, 41)(25, 63)(27, 64)(29, 67)(30, 54)
    (31, 52)(33, 60)(34, 70)(37, 57)(39, 47)(40, 48)(43, 68)(49,
    65)(51, 55)(53, 71)(56, 59)(58, 66)(62, 72);
```

Similarly since we check as to whether or not NL[2] is $U_3(3)$, we need to find the two generators of NL[2].

```
>for g,h in NL[2] do if Order(g) eq 2 and Order(h) eq 6 and
for | if>NL[2] eq sub<NL[2]|g,h>
for | if>then a:=g; b:=h; break; end if;end for;

A:=G1!(1, 21)(3, 25)(4, 27)(6, 44)(9, 55)(11, 60)(13, 22)(14,
    53)(15, 63)(17, 64)(18, 30)(19, 43)(20, 70)(23, 68)(28, 36)
    (31, 59)(33, 35)(37, 72)(41, 58)(42, 65)(46, 56)(47, 62)
    (48, 66)(50, 52);
B:=G1!(1, 23, 62, 5, 59, 40)(2, 49, 66)(3, 50, 39)(4, 33, 43,
    25, 53, 14)(6, 61, 56, 26, 72, 48)(7, 51, 64, 21, 29, 41)
    (8, 68, 71)(9, 37, 16, 19, 54, 12)(10, 70, 22)(11, 57, 47,
    17, 32, 38)(13, 31, 18)(15, 63, 65, 60, 58, 20)(24, 35, 45,
    44, 55, 34)(27, 67, 28)(30, 42, 69)(36, 52, 46);
```

Letting i and j represent A and B respectively, we run the Schreier System in order to find the action of R on the generators of NL[2], A and B.

```
> AA:=[Id(NN) : i in [1..2]];
> for i in [1..6048] do if A^R eq ArrayP[i] then AA[1]:=Sch[i];
  Sch[i]; end if; end for;
i * j * i * j * i * j^3 * i * j^-1 * i * j^-1 * i
>
> for i in [1..6048] do if B^R eq ArrayP[i] then AA[1]:=Sch[i];
  Sch[i]; end if; end for;
i * j * i * j^-2 * i * j^-1 * i * j^-2 * i * j^-1 * i * j^-1 *
i * j
```

Finally we let k represent the generator of CyclicGroup(2) so we can add it and the relations above to our presenation. We note that $i^k = i*j*i*j*i*j^3*i*j^{-1}*i*j^{-1}*i$, and $j^k = i*j*i*j^{-2}*i*j^{-1}*i*j^{-2}*i*j^{-1}*i*j^{-1}*i*j$. So now we can coomplete the presentation.

```
> H<i,j,k>:=Group<i,j,k|i^2,j^6,(i*j)^7,(i,(i*j^2)^3),j^3*(j^2,
  i*j^3*i)^2,
> k^2,i^k=i * j * i * j * i * j^3 * i * j^-1 * i * j^-1 * i,
> j^k=i * j * i * j^-2 * i * j^-1 * i * j^-2 * i * j^-1 * i *
  j^-1 * i * j>;
> f1,h,k1:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(NL[3],h);
> s;
true
```

As we can see above, the presentation we derived from the composition factors proved to be isomorphic to G giving us the isomorphic type $U_3(3):2$.

3.3 Extension Problem

Consider the group

$$G =< a, b, x, t|a^4, b^4, (a, b), x^2, a^x = b, b^x = a, t^2, (t, b), (x * t)^0, (x * a * t * t^x)^3,$$
$$(a * t * t^a)^0, (a * x * t^(x^2) * t^x * t)^0, (a * x * t^(x^2) * t^a * t)^{10} >$$

Using MAGMA we have the following composition factors

```
CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  Alternating(6)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    1
```

So we have the composition series $G_1 \supseteq G_2 \supseteq G_3 \supseteq G_4 \supseteq 1$ where $G = (G_1/G_2) \cdot (G_2/G_3) \cdot (G_3/G_4) \cdot (G_4/1)$
$G = C_2 \cdot A_6 \cdot C_3 \cdot C_2$.

The group also has the following normal lattice.

```
NL:=NormalLattice(G1);
NL;

Normal subgroup lattice
-----------------------

[9]  Order 4320  Length 1  Maximal Subgroups: 6 7 8
---
[8]  Order 2160  Length 1  Maximal Subgroups: 5
[7]  Order 2160  Length 1  Maximal Subgroups: 5
[6]  Order 2160  Length 1  Maximal Subgroups: 4 5
---
[5]  Order 1080  Length 1  Maximal Subgroups: 3
```

```
---
[4]  Order 6     Length 1  Maximal Subgroups: 2 3
---
[3]  Order 3     Length 1  Maximal Subgroups: 1
[2]  Order 2     Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups:
```

After investigating the normal lattice we devised two methods to solve the extension problem.

3.3.1 Method 1: $2 \times [A_6 \bullet 3 : 2]$

First we check the normal lattice for a central extension and we also check to see if there is a direct product between the normal subgroups.

```
> Center(G1);
Permutation group acting on a set of cardinality 540
Order = 2
> for i in [6..8] do if IsIsomorphic(DirectProduct(NL[2],NL[i]),G1)
for|if> then i; end if; end for;
7
8
```

Now we were given 7 and 8 are direct products. We chose to investigate NL[8]. Since NL[2] is a normal subgroup of order 2, there is a direct product between NL[2] and NL[8]. Now we look at the composition factors and the normal lattice of 8.

```
> CompositionFactors(NL[8]);
    G
    |  Cyclic(2)
    *
    |  Alternating(6)
    *
    |  Cyclic(3)
    1
> nl:=NormalLattice(NL[8]);
> nl;
Normal subgroup lattice
-----------------------

[4]  Order 2160  Length 1  Maximal Subgroups: 3
```

```
---
[3]  Order 1080  Length 1  Maximal Subgroups: 2
---
[2]  Order 3     Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups:
```

There is a cyclic(3) in the composition factors of NL[8] and a group in the normal lattice of NL[8] that is of order 3. Alternating(6) is of order 360 and is not a normal subgroup of NL[8], therefore it is not a direct product with cyclic(3). Now we look to see if [3] is a semidirect product of cyclic(3) by Alternating(6), however NL[8] does not have a subgroup of order 360.

Since NL[3] is not a direct product not a semi-direct product of Alternating(6) by cyclic(3) we check the derived groups of NL[8]. The derived group of NL[8] is of order 1080 which is nl[3] of NL[8]. We check that nl[3] is perfect, which we find is true. Now we know that cyclic(3) and alternating(6) is a perfect group of order 1080 and we can get the presentation.

```
> D:=DerivedGroup(NL[8]);
> D;
Permutation group D acting on a set of cardinality 540
Order = 1080 = 2^3 * 3^3 * 5
> CompositionFactors(D);
    G
    |  Alternating(6)
    *
    |  Cyclic(3)
    1
```

Now we use the perfect group database to find the presentation of the perfect group.

```
> DB:= PerfectGroupDatabase();
> DB;
Database of Perfect Groups
> for i in [1..2] do Group(DB,1080,i); end for;
Finitely presented group on 3 generators
Relations
    a^6 = Id($)
    b^3 = Id($)
```

```
    c^3 = Id($)
    (b * c)^4 = Id($)
    (b * c^-1)^5 = Id($)
    a^-1 * b^-1 * c * b * c * b^-1 * c * b * c^-1 = Id($)
[
    <[ 18 ], [
        Finitely presented group on 2 generators (free)
    ]>
]
> s:=IsIsomorphic(PermutationGroup(DB,1080,1),nl[3]);
> s;
true
> H<a,b,c>:=Group<a,b,c|a^6,b^3,c^3,(b*c)^4,(b*c^-1)^5,
a^-1*b^-1 * c * b * c *  b^-1 * c * b * c^-1>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> #H1;
1080
> s,t :=IsIsomorphic(H1,nl[3]);
> s;
true
```

Cyclic(2) is left in NL[8]. Since there is no normal group of order 2 in NL[8] there is not a direct product. There is a subgroup of order 2 in NL[8] so it is a semidirect product with Alternating(6) and Cyclic(3). Now that we have the presentation of Cyclic(3) and Alternating(6) we need to find the action of Cyclic(2) on the presentation by using the schreier system. And finally, we can add the action of Cyclic(2) into the presentation.

```
HHG<a,b,c,u,d>:=Group<a,b,c,u,d|a^6,b^3,c^3,(b*c)^4,(b*c^-1)^5,
a^-1*b^-1*c*b*c*b^-1*c*b*c^-1, u^2, a^u=b*a^-1*b^-1,
b^u=a*c^-1*a, c^u=c^-1*b^-1*c, d^2,(d,a),(d,b),(d,c),(d,u)>;
```

Lastly we check that our presentation is isomorphic to our extention problem which we find to be true. So the isomorphic image we discover from Method 1 is $2 \times [(A_6 \bullet 3) : 2]$.

3.3.2 Method 2: $6 \bullet PGL_2(9)$

As before, the extension problem has the following normal lattice.

```
NL:=NormalLattice(G1);
```

```
NL;

Normal subgroup lattice
-----------------------

[9]  Order 4320  Length 1  Maximal Subgroups: 6 7 8
---
[8]  Order 2160  Length 1  Maximal Subgroups: 5
[7]  Order 2160  Length 1  Maximal Subgroups: 5
[6]  Order 2160  Length 1  Maximal Subgroups: 4 5
---
[5]  Order 1080  Length 1  Maximal Subgroups: 3
---
[4]  Order 6     Length 1  Maximal Subgroups: 2 3
---
[3]  Order 3     Length 1  Maximal Subgroups: 1
[2]  Order 2     Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups:
```

When we investigate NL[4] we see that NL[4] is a cyclic group of order 6 generated by D. We check as to whether or not we will have PGL(2,9) when we factor G by NL[4], which we will call q. We check if q is isomorphic to PGL(2,9) and find that they are isomorphic to each other.

```
> NL[4];
Permutation group acting on a set of cardinality 540
Order = 6 = 2 * 3
 > Order(NL[4].1);
6
 > D:=NL[4].1;
 > NL[4] eq sub<NL[4]|D>;
true
 > IsCyclic(NL[4]);
true
 > q:=quo<G1|NL[4]>;
 > P:=PGL(2,9);
 > s:=IsIsomorphic(q,P);
 > s;
true
 > #q;
720
```

Now we need to find the presentation of q. We use FPGoup(q) to find the finitely presented group of q.

```
>  FPGroup(q);
Finitely presented group on 4 generators
Relations
  $.1^2 = Id($)
  $.2^2 = Id($)
  $.3^2 = Id($)
  $.4^2 = Id($)
  ($.1 * $.2)^2 = Id($)
  $.1 * $.3 * $.2 * $.3 = Id($)
  ($.2 * $.4)^2 = Id($)
  $.4 * $.1 * $.4 * $.2 * $.1 * $.4 * $.1 * $.4 * $.1 = Id($)
  ($.1 * $.4 * $.3 * $.4)^3 = Id($)
  $.4 * $.3 * $.1 * $.4 * $.3 * $.4 * $.3 * $.1 * $.4 * $.3 *
$.2 * $.4 * $.3 * $.4 * $.1 * $.3 * $.4 * $.3 * $.4 * $.3=Id($)
```

Using the relations given and allowing \$.1, \$.2, \$.3, and \$.4 to be g, h, i, and j respectively, we find the presentation of q, denoted by H, to be the following.

```
>  H<g,h,i,j>:=Group<g,h,i,j|g^2,h^2,i^2,j^2, (g*h)^2,(h*j)^2,
g * i * h * i, j * g * j * h * g * j * g * j * g,
(g * j * i * j)^3, j * i * g * j *i * j * i * g * j * i * h *
j * i* j* g * i * j * i * j * i>;
```

We also confirm that the order of H is 720, which is the same as the order of q, then we check to see that H is isomorphic to q.

```
>  f1,h1,k1:=CosetAction(H,sub<H|Id(H)>);
> s:=IsIsomorphic(q,h1);
> s;
true
```

Next we will look at the transversals of NL[4]. Now let T2, T3, T4, and T5 represent T[2],T[3], T[4], and T[5] respectively. T[1] is the identity and as such does not need to be represented as T1. We will then name them each g, h, i, and j respectively.

```
> T2:=T[2]; T3:=T[3];T4:=T[4];T5:=T[5];
> /* G/NL[4]= <NT2,NT3,NT4,NT5>; */
> g:=T[2]; h:=T[3]; i:=T[4]; j:=T[5];
```

We checked if the order of the relations given above matched the order of the relations when replaced by the transversals. We learn that $(j*g*j*h*g*j*g*j*g) = D^3$ and $(j*i*g*j*i*j*i*g*j*i*h*j*i*j*g*i*j*i*j*i) = D^4$. Next we want to find the action of q on D. To do so we use the following loops.

```
> for m in [1..6] do if D^g eq D^m then m; end if; end for;
1
> for m in [1..6] do if D^h eq D^m then m; end if; end for;
1
> for m in [1..6] do if D^i eq D^m then m; end if; end for;
1
> for m in [1..6] do if D^j eq D^m then m; end if; end for;
5
```

Therefore we note that $D^g = D$, $D^h = D$, $D^i = D$, $D^j = D^5$. So now we have the presentaion of D and q. We check that the order of G is equal to the order of our extension problem.

```
> G<d,g,h,i,j>:=Group<d,g,h,i,j|d^6,g^2,h^2,i^2,j^2,(g*h)^2,
>(h*j)^2, g*i*h*i, j*g*j*h*g*j*g*j*g=d^3, (g*j*i*j)^3,
> j*i*g*j*i*j*i*g*j*i*h*j*i*j*g*i*j*i*j*i=d^4,
> d^g=d,d^h=d,d^i=d,d^j=d^5>;
> #G;
4320
> f2,g2,k2:=CosetAction(G,sub<G|Id(G)>);
> #g2;
4320
> s:=IsIsomorphic(G1,g2);
> s;
true
```

As we can see above, the presentation we derived from the composition factors proved to be isomorphic to our extension problem. So the isomorphic image we get here is $6 \bullet PGL_2(9)$.

Chapter 4

Wreath Product Progenitors

4.1 Preliminaries

Definition 4.1 (Wreath Product)**.** *Let H and K be permutation groups on X and Y respectively then $Z = X \times Y$ where $Z = \{(x, y)|x \in X, y \in Y\}$ and $X \cap Y = \emptyset$. If a permutation group is defined on Z then we call this the Wreath Product of H by K, denoted $H \wr K$. Define $H \leq S_X$ and $K \leq S_Y$.*

Definition 4.2. *Let $\gamma \in H$ and y be a fixed element of Y defined $\gamma(y) : \{(x, y) \mapsto ((x)\gamma, y), (x, y_1) \mapsto (x, y_1)$ if $y_1 \neq y\}$.*

Definition 4.3. *If $k \in K$ then $k^* : (x, y) \to (x, (y)k)$.*

Definition 4.4. *If $\gamma(y)$ and k^* are permutations of S_Z then*

$$\phi : H \to S_Z \text{ given by } \gamma \to \gamma(y),$$

where ϕ is 1-1 and a homomorphism. Then the image

$$\phi(H) = \{\gamma(y)|\gamma \in H\} = H(y).$$
$$\text{And } \psi : K \to S_Z \text{ given by } k \to k^*,$$

where ψ is also 1-1 and a homomorphism. Then $\psi(k) = k^* = \{k^*|k \in K\}$.

Theorem 4.5. *The wreath product of the group H by K, denoted $H \wr K$ is the semi-direct product $H^n : K$, where n is the number of letters on which K acts and H^n is the direct copies of H. K permutes the n isomorphic copies of H.*

$$H \wr K = \{H(y), K^* | y \in Y\}$$
$$H \wr K = \prod H(y) : K^*$$

Definition 4.6 (Base)**.** *The base of the wreath product is the direct copies of H, denoted* $\prod H(y)$*.*

4.2 Construction of the Wreath Product $2^{*8} : \mathbb{Z}_4 \wr \mathbb{Z}_2$

Let H and K be permutation groups on X and Y, respectively. The wreath product of the group H by K, denoted by $H \wr K$, is the semi-direct product $H^n : K$, where n is the number of letters on which K acts and H^n is the direct product of n copies of H. Then K permutates the n copies of H for a semi-direct product. Let $Z = \{(x, y)|x \in X, y \in Y\}$. We will define a permuation group on Z, the wreath product of H by K.

To show an example we will find the wreath product $\mathbb{Z}_4 \wr \mathbb{Z}_2$.
Let $H = \mathbb{Z}_7 =< e, (1, 2, 3, 4) >$ and $K = \mathbb{Z}_2 =< e, (5, 6) >$, where $X = \{1, 2, 3, 4\}$ and $Y = \{5, 6\}$. Now $Z = \{(x, y)|x \in X, y \in Y\}$. We can easily find that $Z = \{(1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6), (4, 5), (4, 6)\}$.

Let $\gamma \in H$ and y be a fixed element of Y. Define $\gamma(y) : \{(x, y) \mapsto ((x)\gamma, y), (x, y_1) \mapsto (x, y_1)$ if $y_1 \neq y\}$.

Table 4.1: Mapping of $\gamma(5)$

1	(1,5)	$\to$	(2,5)	3
2	(1,6)	$\to$	(1,6)	2
3	(2,5)	$\to$	(3,5)	5
4	(2,6)	$\to$	(2,6)	4
5	(3,5)	$\to$	(4,5)	7
6	(3,6)	$\to$	(3,6)	6
7	(4,5)	$\to$	(1,5)	1
8	(4,6)	$\to$	(4,6)	8

So $\gamma(5) = (1, 3, 5, 7)$ and similarly we find $\gamma(6) = (2, 4, 6, 8)$. Now let $k \in K$ and define $k^* : (x, y) \mapsto (x, (y)k)$.

Table 4.2: Mapping of $k^* = (5,6)^*$

1	(1,5)	$\rightarrow$	(1,6)	2
2	(1,6)	$\rightarrow$	(1,5)	1
3	(2,5)	$\rightarrow$	(2,6)	4
4	(2,6)	$\rightarrow$	(2,5)	3
5	(3,5)	$\rightarrow$	(3,6)	6
6	(3,6)	$\rightarrow$	(3,5)	5
7	(4,5)	$\rightarrow$	(4,6)	8
8	(4,6)	$\rightarrow$	(4,5)	7

So we see that $k^* = (1,2)(3,4)(5,6)(7,8)$. We know that the presentation for $\mathbb{Z}_4^2 =< a|a^4 > \times < b|b^4 >$ and the presentation for $\mathbb{Z}_2 =< u|u^2 >$. Let $a = (1,3,5,7)$, $b = (2,4,6,8)$, and $u = (1,2)(3,4)(5,6)(7,8)$. Finding the action of $\mathbb{Z}_2$ on $\mathbb{Z}_4$ in words we get:

$$a^x = b,\ b^x = a$$

Using the information we gathered we can now find the wreath product presentation:

$$\mathbb{Z}_4 \wr \mathbb{Z}_2 =< a, b, u|a^4, b^4, (a,b), u^2, a^u = b, b^u = a >$$

We confirm this presentation is isomorphic to the wreath product using MAGMA. From here we introduce a new variable t, let $t = t_1$ and we note that the stabilizer of 1 in $\mathbb{Z}_4 \wr \mathbb{Z}_2$ is

$$< (2,4,6,8) >$$

Thus t commutes with e, b, b^2, b^3.

Finally we find that the presentation for the progenitor is

$$2^8 : \mathbb{Z}_4 \wr \mathbb{Z}_2 =< a, b, u, t|a^4, b^4, (a,b), u^2, a^u = b, b^u = a, t^2, (t,b) >$$

By adding the following relations to the progenitor we get the images given below.

$$(u*t)^i, (u*a*t*t^u)^j, (a*t*t^a)^k, (a*u*t^{u^2}*t^u*t)^l, (a*u*t^{u^2}*t^a*t)^m$$

Table 4.3: Homomorphic Images of $2^{*8} : \mathbb{Z}_4 \wr \mathbb{Z}_2$

i	j	k	l	m	Index of G	Order of G	Isomorphic Type
0	2	0	8	0	96	3072	$2^{2^\bullet}[2^{2^\bullet}(2^3 : S_4)]$
0	3	0	0	9	612	4896	$2 \times L_2(17)$
0	3	0	0	10	540	4320	$2 \times [A_6 : 3]$
0	3	0	4	0	1224	9792	$2 \times 2 \times L_2(17)$
0	3	0	5	0	3720	29760	$2 \times L_2(31)$
0	5	2	0	3	1950	15600	$2 \times L_2(25)$
4	0	0	10	6	7200	230400	$2^\bullet[2^\bullet[2 \times (A_5 \times A_5) : 2^2 : 2]]$
4	0	2	0	6	162	1296	$2 \times [3^2 : 3^2 : 2^2 : 2]$
4	0	5	10	0	16200	518400	$2 \times [(A_6 \times A_6) : 2]$
4	0	6	10	6	1800	57600	$2^\bullet[(A_5 \times A_5) : 2 : 2^2]$
5	0	2	7	0	3045	24360	$PGL_2(29)$
6	0	2	0	3	6552	52416	$2^\bullet[(2^3 \times L_2(13)) : (3 : 2)]$
6	0	2	7	3	1638	13104	$2^\bullet[(3 \times L_2(13)) : 2]$

Table 4.4: Progenitors of $2^{*8} : \mathbb{Z}_4 \wr \mathbb{Z}_2$ where $Kernel > 1$

i	j	k	l	m	Index	Order of G	Kernel
0	0	0	4	2	2184	34944	2
0	8	4	0	2	576	9216	2
0	10	0	10	2	660	10560	2
7	0	0	4	0	546	8736	2

4.3 Wreath Product Progenitor $2^{*10} : \mathbb{Z}_2 \wr S_5$

For another example we will find the wreath product $\mathbb{Z}_2 \wr S_5$.
Let $H = \mathbb{Z}_2 =< e, (1,2) >$ and $K = S_5 =< (3,4,5,6,7), (3,4) >$, where $X = \{1,2\}$ and $Y = \{3,4,5,6,7\}$. Now $Z = \{(x,y) | x \in X, y \in Y\}$. We can easily find that $Z = \{(1,3), (1,4), (1,5), (1,6), (1,7), (2,3), (2,4), (2,5), (2,6), (2,7)\}$.

Let $\gamma \in H$ and y be a fixed element of Y. Define $\gamma(y) : \{(x,y) \mapsto ((x)\gamma, y), (x, y_1) \mapsto (x, y_1)$ if $y_1 \neq y\}$.

Table 4.5: Mapping of $\gamma(3)$

1	(1,3)	→	(2,3)	6
2	(1,4)	→	(1,4)	2
3	(1,5)	→	(1,5)	3
4	(1,6)	→	(1,6)	4
5	(1,7)	→	(1,7)	5
6	(2,3)	→	(1,3)	1
7	(2,4)	→	(2,4)	7
8	(2,5)	→	(2,5)	8
9	(2,6)	→	(2,6)	9
10	(2,7)	→	(2,7)	10

So $\gamma(3) = (1,6)$, and similarly we find $\gamma(4) = (2,7)$, $\gamma(5) = (3,8)$, $\gamma(6) = (4,9)$, and $\gamma(7) = (5,10)$. Now let $k \in K$ and define $k^* : (x,y) \mapsto (x, (y)k)$.

Table 4.6: Mapping of $k^* = (3,4,5,6,7)^*$

1	(1,3)	→	(1,4)	2
2	(1,4)	→	(1,5)	3
3	(1,5)	→	(1,6)	4
4	(1,6)	→	(1,7)	5
5	(1,7)	→	(1,3)	1
6	(2,3)	→	(2,4)	7
7	(2,4)	→	(2,5)	8
8	(2,5)	→	(2,6)	9
9	(2,6)	→	(2,7)	10
10	(2,7)	→	(2,3)	6

So $k_1^* = (1,2,3,4,5)(6,7,8,9,10)$ and $k_2^* = (1,2)(6,7)$. We know that the presentation for $\mathbb{Z}_2^5 =< a|a^2 > \times < b|b^2 > \times < c|c^2 > \times < d|d^2 > \times < e|e^2 >$ and the presentation for $S_5 =< u, v|u^5, v^2, (uv)^4, (u, v)^3 >$. Let $a = (1, 6)$, $b = (2, 7)$, $c = (3, 8)$, $d = (4, 9)$, $e = (5, 10)$, $u = (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)$, and $v = (1, 2)(6, 7)$. Finding the action of S_5 on $\mathbb{Z}_2$ in words we get:

$$a^u = b,\ b^u = c,\ c^u = d,\ d^u = e,\ e^u = a$$
$$a^v = b,\ b^v = a,\ c^v = c,\ d^v = d,\ e^v = e$$

Using the information we gathered we can now find the wreath product presentation:

$$\mathbb{Z}_2 \wr S_5 =< a, b, c, d, e, u, v|a^2, b^2, c^2, d^2, e^2, (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e), u^5, v^2, (uv)^4, (u, v)^3, a^u = b, b^u = c, c^u = d, d^u = e, e^u = a, a^v = b, b^v = a, c^v = c, d^v = d, e^v = e >$$

We confirm this presentation is isomorphic to the wreath product using MAGMA. From here we introduce a new variable t, let $t = t_1$ and we note that the stabilizers of 1 in $\mathbb{Z}_2 \wr S_5$ are

(2,7)
(3,8)
(4,9)
(5,10)
(4,10)(5,9)
(3,4,10)(5,8,9)
(2,10,9)(4,7,5)
(2,10)(5,7)

Thus t commutes with $b, c, d,$ and e, however we need to use the Schreier System in MAGMA to find that t also commutes with $(d*e*u^2*v*u^{-2})$, $(d*e*v*u*v^{-1}*u*v)$, $(b*d*v*u^{-2}*v*u)$, $(b*e*v*u*v*u^{-1}*v)$.

Finally we find that the presentation for the progenitor is

$$2^{10} : \mathbb{Z}_2 \wr S_5 =< a, b, c, d, e, u, v, t|a^2, b^2, c^2, d^2, e^2, (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e), u^5, v^2, (uv)^4, (u, v)^3, a^x = b, b^x = c, c^x = d, d^x = e, e^x = a, a^y = b, b^y = a, c^y = c, d^y = d, e^y = e, t^2, (t, b), (t, c), (t, d), (t, e), (t, d*e*u^2*v*u^{-2}), (t, d*e*v*u*v^{-1}*u*v), (t, b*d*v*u^{-2}*v*u), (t, b*e*v*u*v*u^{-1}*v) >$$

By adding the following relations to the progenitor we get the images given below.

$(a*t^a)^i, (u*a*t*t^u)^j, (v*a*t*t^v)^k, (a*u*v*t^{u^2}*t^u*t^v)^l, (a*u*v*t^{u^2}*t^a*t)^m$

Table 4.7: Homomorphic Images of $2^{*10} : \mathbb{Z}_2 \wr S_5$

i	j	k	l	m	Index	Order of G	Isomorphic Type
0	4	0	0	0	768	2,949,120	$2^{\bullet}[2^{11} : A_6 : 2]$
3	0	0	0	8	243	933,120	$3^5 : 2^5 : A_5 : 2$

Table 4.8: Progenitors of $2^{*10} : \mathbb{Z}_2 \wr S_5$ where $Kernel > 1$

i	j	k	l	m	Index	Order of G	Kernel
0	0	0	0	5	6	1440	2
0	0	2	0	8	1040	245,760	2
0	4	0	0	10	380	1,474,560	32
0	5	0	0	8	32	122,880	32

4.4 Wreath Product Progenitor $2^{*12} : \mathbb{Z}_3 \wr S_4$

For another example we will find the wreath product $\mathbb{Z}_3 \wr S_4$.
Let $H = \mathbb{Z}_3 =< e, (1,2,3) >$ and $K = S_5 =< (4,5,6,7), (4,5) >$, where $X = \{1,2,3\}$ and $Y = \{4,5,6,7\}$. Now $Z = \{(x,y)|x \in X, y \in Y\}$. We can easily find that $Z = \{(1,4),(1,5),(1,6),(1,7),(2,4),(2,5),(2,6),(2,7),(3,4),(3,5),(3,6),(3,7)\}$.

Let $\gamma \in H$ and y be a fixed element of Y. Define $\gamma(y) : \{(x,y) \mapsto ((x)\gamma, y), (x, y_1) \mapsto (x, y_1)$ if $y_1 \neq y\}$.

So $\gamma(4) = (1,5,9)$, and similarly we find $\gamma(5) = (2,6,10)$, $\gamma(6) = (3,7,11)$, and lastly $\gamma(7) = (4,8,12)$. Now let $k \in K$ and define $k^* : (x,y) \mapsto (x,(y)k)$.

So $k_1^* = (1,2,3,4)(5,6,7,8)(9,10,11,12)$ and $k_2^* = (1,2)(5,6)(9,10)$. We know that the presentation for $\mathbb{Z}_3^4 =< a|a^3 > \times < b|b^3 > \times < c|c^3 > \times < d|d^3 >$ and the presentation for $S_5 =< u, v|u^4, v^2, (uv)^3 >$. Let $a = (1,5,9)$, $b = (2,6,10)$, $c = (3,7,11)$, $d = (4,8,10)$, $u = (1,2,3,4)(5,6,7,8)(9,10,11,12)$, and $v = (1,2)(5,6)(9,10)$. Finding the action of S_4 on $\mathbb{Z}_3$ in words we get:

Table 4.9: Mapping of $\gamma(4)$

1	(1,4)	$\rightarrow$	(2,4)	5
2	(1,5)	$\rightarrow$	(1,5)	2
3	(1,6)	$\rightarrow$	(1,6)	3
4	(1,7)	$\rightarrow$	(1,7)	4
5	(2,4)	$\rightarrow$	(3,4)	9
6	(2,5)	$\rightarrow$	(2,5)	6
7	(2,6)	$\rightarrow$	(2,6)	7
8	(2,7)	$\rightarrow$	(2,7)	8
9	(3,4)	$\rightarrow$	(1,4)	1
10	(3,5)	$\rightarrow$	(3,5)	10
11	(3,6)	$\rightarrow$	(3,6)	11
12	(3,7)	$\rightarrow$	(3,7)	12

Table 4.10: Mapping of $k^* = (4, 5, 6, 7)^*$

1	(1,4)	$\rightarrow$	(1,5)	5
2	(1,5)	$\rightarrow$	(1,6)	2
3	(1,6)	$\rightarrow$	(1,7)	3
4	(1,7)	$\rightarrow$	(1,4)	4
5	(2,4)	$\rightarrow$	(2,5)	9
6	(2,5)	$\rightarrow$	(2,6)	6
7	(2,6)	$\rightarrow$	(2,7)	7
8	(2,7)	$\rightarrow$	(2,4)	8
9	(3,4)	$\rightarrow$	(3,5)	1
10	(3,5)	$\rightarrow$	(3,6)	10
11	(3,6)	$\rightarrow$	(3,7)	11
12	(3,7)	$\rightarrow$	(3,4)	12

$$a^u = b,\ b^u = c,\ c^u = d,\ d^u = a$$
$$a^v = b,\ b^v = a,\ c^v = c,\ d^v = d$$

Using the information we gathered we can now find the wreath product presentation:

$$\mathbb{Z}_3 \wr S_4 =< a, b, c, d, u, v | a^3, b^3, c^3, d^3, (a, b), (a, c), (a, d), (b, c), (b, d), (c, d), u^4, v^2, (uv)^3,$$
$$a^u = b, b^u = c, c^u = d, d^u = a, a^v = b, b^v = a, c^v = c, d^v = d >$$

We confirm this presentation is isomorphic to the wreath product using MAGMA. From here we introduce a new variable t, let $t = t_1$ and we note that the stabilizers of 1 in $\mathbb{Z}_3 \wr S_4$ are

(2,6,10)
(3,7,11)
(4,8,10)
(2,3)(4,12)(6,7)(10,11)
(2,3,12,10,11,8,6,7,4)
(2,4,10,12,6,8)

Thus t commutes with e, b, b^2, c, c^2, d and d^2, however we need to use the Schreier System in MAGMA to find that t also commutes with $(d^{-1} * u^{-1} * v * u)$, $(u * v * d^{-1})$, $(v * u^2 * v * u * b^{-1})$.
Finally we find that the presentation for the progenitor is

$$2^{12} : \mathbb{Z}_3 \wr S_4 =<$$
$$a, b, c, d, u, v, t | a^3, b^3, c^3, d^3, (a, b), (a, c), (a, d), (b, c), (b, d), (c, d), u^4, v^2, (uv)^3,$$
$$a^u = b, b^u = c, c^u = d, d^u = a, a^v = b, b^v = a, c^v = c, d^v = d, t^2, (t, b), (t, c), (t, d),$$
$$(t, d^{-1} * u^{-1} * v * u), (t, u * v * d^{-1}), (t, v * u^2 * v * u * b^{-1}) >$$

By adding the following relations to the progenitor we get the images given below.

$$(a * t^a)^i, (u * a * t * t^u)^j, (v * a * t * t^v)^k, (a * u * v * t^{a^2} * t^u * t^v)^l, (a * u * v * t^{u^2} * t^a * t)^m$$

4.5 Wreath Product Progenitor $2^{*14} : \mathbb{Z}_7 \wr \mathbb{Z}_2$

For another example we will find the wreath product $\mathbb{Z}_7 \wr \mathbb{Z}_2$.
Let $H = \mathbb{Z}_7 =< e, (1, 2, 3, 4, 5, 6, 7) >$ and $K = \mathbb{Z}_2 =< e, (8, 9) >$, where $X =$

Table 4.11: Homomorphic Images of $2^{*12} : \mathbb{Z}_3 \wr S_4$

i	j	k	l	m	Index	Order of G	Isomorphic Type
0	0	2	6	7	91	2184	$PGL_2(13)$
0	0	6	0	4	10	240	$2 \times S_5$
0	0	6	3	6	256	497664	$2^8 : [(3^3 \times 3) : S_4]$

Table 4.12: Progenitors of $2^{*12} : \mathbb{Z}_3 \wr S_4$, where $Kernel > 1$

i	j	k	l	m	Index	Order of G	Kernel
0	3	0	0	0	14	336	3

$\{1, 2, 3, 4, 5, 6, 7\}$ and $Y = \{8, 9\}$. Now $Z = \{(x, y)|x \in X,\ y \in Y\}$. We can easily find that $Z = \{(1, 8), (1, 9), (2, 8), (2, 9), (3, 8), (3, 9), (4, 8), (4, 9), (5, 8)(5, 9), (6, 8), (6, 9), (7, 8), (7, 9)\}$. Let $\gamma \in H$ and y be a fixed element of Y. Define $\gamma(y) : \{(x, y) \mapsto ((x)\gamma, y), (x, y_1) \mapsto (x, y_1)$ if $y_1 \neq y\}$.

Table 4.13: Mapping of $\gamma(8)$

1	(1,8)	$\rightarrow$	(2,8)	3
2	(1,9)	$\rightarrow$	(1,9)	2
3	(2,8)	$\rightarrow$	(3,8)	5
4	(2,9)	$\rightarrow$	(2,9)	4
5	(3,8)	$\rightarrow$	(4,8)	7
6	(3,9)	$\rightarrow$	(3,9)	6
7	(4,8)	$\rightarrow$	(5,8)	9
8	(4,9)	$\rightarrow$	(4,9)	8
9	(5,8)	$\rightarrow$	(6,8)	11
10	(5,9)	$\rightarrow$	(5,9)	10
11	(6,8)	$\rightarrow$	(7,8)	13
12	(6,9)	$\rightarrow$	(6,9)	12
13	(7,8)	$\rightarrow$	(1,8)	1
14	(7,9)	$\rightarrow$	(7,9)	14

So $\gamma(8) = (1, 3, 5, 7, 9, 11, 13)$ and similarly we find $\gamma(9) = (2, 4, 6, 8, 10, 12, 14)$. Now let $k \in K$ and define $k^* : (x, y) \mapsto (x, (y)k)$.

So we see that $k^* = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)$. We know that the presentation for $\mathbb{Z}_7^2 =< a|a^7 > \times < b|b^7 >$ and the presentation for $\mathbb{Z}_2 =< u|u^2 >$. Let $a = (1, 3, 5, 7, 9, 11, 13)$, $b = (2, 4, 6, 8, 10, 12, 14)$, and $u = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)$

Table 4.14: Mapping of $k^* = (8,9)^*$

1	(1,8)	$\rightarrow$	(1,9)	2
2	(1,9)	$\rightarrow$	(1,8)	1
3	(2,8)	$\rightarrow$	(2,9)	4
4	(2,9)	$\rightarrow$	(2,8)	3
5	(3,8)	$\rightarrow$	(3,9)	6
6	(3,9)	$\rightarrow$	(3,8)	5
7	(4,8)	$\rightarrow$	(4,9)	8
8	(4,9)	$\rightarrow$	(4,8)	7
9	(5,8)	$\rightarrow$	(5,9)	10
10	(5,9)	$\rightarrow$	(5,8)	9
11	(6,8)	$\rightarrow$	(6,9)	12
12	(6,9)	$\rightarrow$	(6,8)	11
13	(7,8)	$\rightarrow$	(7.9)	14
14	(7,9)	$\rightarrow$	(7,8)	13

$(13, 14)$. Finding the action of $\mathbb{Z}_2$ on $\mathbb{Z}_7$ in words we get:

$$a^u = b,\ b^u = a$$

Using the information we gathered we can now find the wreath product presentation:

$$\mathbb{Z}_7 \wr \mathbb{Z}_2 =< a, b, u | a^7, b^7, (a, b), u^2, a^u = b, b^u = a >$$

We confirm this presentation is isomorphic to the wreath product using MAGMA. From here we introduce a new variable t, let $t = t_1$ and we note that the stabilizer of 1 in $\mathbb{Z}_7 \wr \mathbb{Z}_2$ is

$$< (2, 4, 6, 8, 10, 12, 14) >$$

Thus t commutes with only e, b, b^2, b^3, b^4, b^5, and b^6.

Finally we find that the presentation for the progenitor is

$$2^{14} : \mathbb{Z}_7 \wr \mathbb{Z}_2 =< a, b, u, t | a^7, b^7, (a, b), u^2, a^u = b, b^u = a, t^2, (t, b) >$$

By adding the following relations to the progenitor we get the images given below.

$$(u * t * a)^i, (u * t^a)^j, (a * t * t^{a^2})^k, (u * a * t * a^2)^l, (a^3 * t^u * t^a)^m$$

Table 4.15: Homomorphic Images of $2^{*14} : \mathbb{Z}_7 \wr \mathbb{Z}_2$

i	j	k	l	m	Index	Order of G	Isomorphic Type
0	0	0	0	3	2054	201684	$7 : [(7^2 \times 7^2) \bullet D_6]$
0	0	0	4	0	28	2744	$7^3 : 2^2 : 2$
0	0	0	5	0	1715	168070	$7 : [7^4 \bullet D_5]$
0	0	0	5	5	245	24010	$7^4 : D_5$
0	0	0	6	3	294	28812	$7^4 : (S_3 \times 2)$
0	3	0	0	0	21	2058	$7^3 : S_3$
0	4	3	0	4	2304	225792	$2^2 : [L_2(7) \times L_2(7)] : 2$
6	4	7	0	6	48672	47698561	$2^{\bullet}[L_2(13) \times L_2(13)] : 2$

4.6 Wreath Product Progenitor $2^{*15} : \mathbb{Z}_5 \wr \mathbb{Z}_3$

For another example we will find the wreath product $\mathbb{Z}_5 \wr \mathbb{Z}_3$. Let $H = \mathbb{Z}_5 =< e, (1,2,3,4,5) >$ and $K = \mathbb{Z}_3 =< e, (6,7,8) >$, where $X = \{1,2,3,4,5\}$ and $Y = \{6,7,8\}$. Now $Z = \{(x,y) | x \in X, y \in Y\}$. We can easily find that $Z = \{(1,6),(1,7),(1,8),(2,6),(2,7),(2,8),(3,6),(3,7),(3,8),(4,6),(4,7),(4,8),(5,6),(5,7),(5,8)\}$.

Let $\gamma \in H$ and y be a fixed element of Y. Define $\gamma(y) : \{(x,y) \mapsto ((x)\gamma, y), (x, y_1) \mapsto (x, y_1)$ if $y_1 \neq y\}$.

Table 4.16: Mapping of $\gamma(6)$

1	(1,6)	$\rightarrow$	(2,6)	4
2	(1,7)	$\rightarrow$	(1,7)	2
3	(1,8)	$\rightarrow$	(1,8)	3
4	(2,6)	$\rightarrow$	(3,6)	7
5	(2,7)	$\rightarrow$	(2,7)	5
6	(2,8)	$\rightarrow$	(2,8)	6
7	(3,6)	$\rightarrow$	(4,6)	10
8	(3,7)	$\rightarrow$	(3,7)	8
9	(3,8)	$\rightarrow$	(3,8)	9
10	(4,6)	$\rightarrow$	(5,6)	13
11	(4,7)	$\rightarrow$	(4,7)	11
12	(4,8)	$\rightarrow$	(4,8)	12
13	(5,6)	$\rightarrow$	(1,6)	1
14	(5,7)	$\rightarrow$	(5,7)	14
15	(5,8)	$\rightarrow$	(5,8)	15

So $\gamma(6) = (1,4,7,10,13)$ and similarly we find $\gamma(7) = (2,5,8,11,14)$ and $\gamma(8) = (3,6,9,12,15)$. Now let $k \in K$ and define $k^* : (x,y) \mapsto (x,(y)k)$.

Table 4.17: Mapping of $k^* = (6,7,8)^*$

1	(1,6)	$\rightarrow$	(1,7)	2
2	(1,7)	$\rightarrow$	(1,8)	3
3	(1,8)	$\rightarrow$	(1,6)	1
4	(2,6)	$\rightarrow$	(2,7)	5
5	(2,7)	$\rightarrow$	(2,8)	6
6	(2,8)	$\rightarrow$	(2,6)	4
7	(3,6)	$\rightarrow$	(3,7)	8
8	(3,7)	$\rightarrow$	(3,8)	9
9	(3,8)	$\rightarrow$	(3,6)	7
10	(4,6)	$\rightarrow$	(4,7)	11
11	(4,7)	$\rightarrow$	(4,8)	12
12	(4,8)	$\rightarrow$	(4,6)	10
13	(5,6)	$\rightarrow$	(5,7)	14
14	(5,7)	$\rightarrow$	(5,8)	15
15	(5,8)	$\rightarrow$	(5,6)	13

So we see that $k^* = (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)$. We know that the presentation for $\mathbb{Z}_5^3 =< a|a^5 > \times < b|b^5 > \times < c|c^5 >$ and the presentation for $\mathbb{Z}_3 =< u|u^3 >$. Let $a = (1,4,7,10,13)$, $b = (2,5,8,11,14)$, $c = (3,6,9,12,15)$, and $u = (1,2,3)(4,5,6)(7,8,9)\ (10,11,12)(13,14,15)$. Finding the action of $\mathbb{Z}_3$ on $\mathbb{Z}_5$ in words we get:

$$a^u = b,\ b^u = c,\ c^u = a$$

Using the information we gathered we can now find the wreath product presentation:

$$\mathbb{Z}_5 \wr \mathbb{Z}_3 =< a,b,c,u|a^5,b^5,c^5,(a,b),(a,c),(b,c),u^3,a^u = b,b^u = c,c^u = a >$$

We confirm this presentation is isomorphic to the wreath product using MAGMA. From here we introduce a new variable t, let $t = t_1$ and we note that the stabilizers of 1 in $\mathbb{Z}_5 \wr \mathbb{Z}_3$ are

$$< (2,5,8,11,14) >$$
$$< (3,6,9,12,15) >$$

Thus t commutes with only $e,b,b^2,b^3,b^4,c,c^2,c^3$ and c^4.

Finally we find that the presentation for the progenitor is

$$2^{15} : \mathbb{Z}_5 \wr \mathbb{Z}_3 =< a, b, c, u, t | a^5, b^5, c^5, (a, b), (a, c), (b, c), u^3, a^u = b, b^u = c, c^u = a, \\ t^2, (t, b), (t, c) >$$

By adding the following relations to the progenitor we get the images given below.

$$(a * t^a)^i, (u * a * t * t^u)^j, (a * t * t^a)^k, (a * u * t^{u^2} * t^u * t)^l, (a * u * t^{u^2} * t^a * t)^m$$

Table 4.18: Homomorphic Images of $2^{*15} : \mathbb{Z}_5 \wr \mathbb{Z}_3$

i	j	k	l	m	Index	Order of G	Isomorphic Type
0	0	0	2	0	16	48	$2^\bullet S_4$
0	0	0	2	4	8	24	S_4
0	0	1	3	0	480	1440	$2^\bullet[A_5 \times (2^2 : 3)]$
0	0	1	4	7	56	168	$L_2(7)$
0	0	1	4	8	256	768	$2^5 : S_4$
0	0	1	4	9	816	2448	$L_2(17)$
0	0	1	4	10	1320	3960	$[3 \times L_2(11)] : 2$
0	0	1	5	8	4960	14880	$L_2(31)$
0	0	1	6	7	364	1092	$L_2(13)$
0	0	1	8	7	3584	10752	$2^6 : L_2(7)$
0	6	1	4	0	5280	15840	$2^2 : [(3 \times L_2(11)) : 2]$

Chapter 5

Monomial Progenitors

In this chapter we will discuss the full automorphism group of $Aut(m^{*n})$. We will prove that $Aut(m^{*n}) = U(m) \wr S_n$, whose order is $(\phi(m))^n \cdot n!$. Afterwards we will show a few examples. Let us first recall some definitions.

5.1 Preliminaries

Definition 5.1 (Direct Product). *Let H and K be subgroups of G. We say that G is a direct product of H by K, denoted $H \times K$, if the following are true.*

1. *H and K are both normal in G*

2. *$H \cap K = 1$*

Definition 5.2 (Semi-Direct Product). *Let H and K be subgroups of G. We say that G is a semi-direct product of H by K, denoted $H : K$, if the following are true.*

1. *H is normal in G*

2. *$H \cap K = 1$*

Definition 5.3 (Wreath Product). *Let D and Q be groups, let Ω be a finite Q-set, let $\{D_w : w \in \Omega\}$ be a family of isomorphic copies of D indexed by Ω and let $K = \Pi_{w\in\Omega} D_w$, where $\Pi_{w\in\Omega} D_w$ denotes the direct product of D_w's, and $D_w \cong D$ for all $w \in \Omega$. Then the wreath product of D by Q denoted by $D \wr Q$ (or by D wr Q), is the semi-direct product* *of K by Q, where Q acts on K by $q \cdot (d_w) = (d_{qw})$ for $q \in Q$ and $(d_w) \in \Pi_{w\in\Omega} D_w$. The normal subgroup K of $D \wr Q$ is called the base of the wreath product. [?]*

5.2 The full automorphism group $Aut(m^{\star n})$.

We first count the number of automorphisms in this group. We fix a t_i, say t_1, and map it to its $\phi(m)$ powers that are relatively prime to m while keeping $t_2, t_3, \ldots, t_n$ fixed. This gives $\phi(m)$ automorphisms of $Aut(m^{\star n})$. Moreover, these $\phi(m)$ automorphisms form a subgroup of $Aut(m^{\star n})$ isomorphic to $U(m)$, the group of positive integers less than m and relatively prime to m under multiplication modulo m. Call this subgroup of $Aut(m^{\star n})$, $U_{t_1}(m)$. Now, each of these $\phi(m)$ automorphisms of $U_{t_1}(m)$ gives another $\phi(m)$ automorphisms when t_2 is mapped to its $\phi(m)$ powers that are relatively prime to m. Thus, we have seen $(\phi(m))^2$ automorphisms in $Aut(m^{\star n})$. We repeat the process with $t_3, t_4, \ldots, t_n$ and obtain $(\phi(m))^n$ automorphisms of $Aut(m^{\star n})$. If we now multiply each of these $(\phi(m))^n$ automorphisms by the n! automorphisms generated by $t_1 \mapsto t_2, t_2 \mapsto t_1,\ t_1 \mapsto t_3, t_3 \mapsto t_1,\ \ldots\ ,\ t_1 \mapsto t_n, t_n \mapsto t_1$, we obtain $(\phi(m))^n \times n!$ automorphisms of $Aut(m^{\star n})$.

We note that the $(\phi(m))^n$ automorphisms of $Aut(m^{\star n})$ give the subgroup $B =< U_{t_1}(m), U_{t_2}(m), \ldots, U_{t_n}(m) >$. We show that $B = \Pi_i U_{t_i}(m)$, a direct product of $U_{t_1}(m), U_{t_2}(m), \ldots, U_{t_n}(m)$. We note that $U_{t_i}(m)^{U_{t_j}(m)} = U_{t_i}(m)$ for $i \neq j$ since, for $a, b \in \{1, 2, \ldots, m-1\}$ and gcd(a,m)=1 and gcd(b,m)=1, $(t_i \mapsto t_i^a)^{(t_j \mapsto t_j^b)} = t_i \mapsto t_i^a$ and $U_{t_i}(m)^{U_{t_i}(m)} = U_{t_i}(m)$. So $U_{t_i}(m)$ is normal in B, for each i. Also, $U_{t_i}(m) \cap < U_{t_j}(m) | i \neq j >= 1$ since an automorphism of the form $t_i \mapsto t_i^a \in < U_{t_j}(m) | i \neq j > \iff a = 1$. Thus, $B = \Pi_i U_{t_i}(m)$, a direct product of $U_{t_1}(m), U_{t_2}(m), \ldots, U_{t_n}(m)$.

Let K be the subgroup of $Aut(m^{\star n})$ generated by the n automorphisms $< t_1 \mapsto t_2, t_2 \mapsto t_1; t_1 \mapsto t_3, t_3 \mapsto t_1; \ldots t_1 \mapsto t_n, t_n \mapsto t_1 >$. This group is isomorphic to S_n, the symmetric group of degree n on $\{1, 2, \ldots, n\}$. We now show that $Aut(m^{\star n}) =< B, K >$ is a semi-direct product of B by K. Let $t_i \mapsto t_i^a \in B$ (a as defined above) and $[t_1 \mapsto t_i, t_i \mapsto t_1] \in K$. Then $(t_i \mapsto t_i^a)^{[t_1 \mapsto t_i, t_i \mapsto t_1]} = t_1 \mapsto t_1^a \in B$ and the automorphisms of B in K permute U_{t_i}'s. Thus B is normal in K. Note that $B \cap K = 1$ since B contains automorphisms of the form $t_i \mapsto t_i^a$ and K contains automorphisms of the form $t_i \mapsto t_j, t_j \mapsto t_i$. Thus B is normal in K and $B \cap K = 1$. Hence, $Aut(m^{\star n}) \cong (B : K) =$

$U_{t_1}(m) \wr K = U(m) \wr S_n$.

5.3 Example $Aut(3^{*2})$

Give the full automorphism group $Aut(3^{*2})$. To find the automorphisms of m^{*n} we first select a t_i to raise to a power of itself that is relatively prime to m. All other t's shall be fixed.

To show the process described we will give an example of finding the group of all monomial automorphisms, $Aut(3^{*2})$. We know that 3^{*2} is on two letters, t_1 and t_2, each of which is of order 3. Now $U(3) = \{1, 2\}$ is a group under multiplication mod 3 and is the set of numbers that are coprime to 3 since $U(n) = \{a | 1 \leq a \leq n, gcd(a, n) = 1\}$. We will now choose a t, say t_1 to send to ${t_1}^1 = t_1$ and fix all other t's. Similarly the second mapping sends $t_1 \rightarrow {t_1}^2$.

So we have two automorphisms. Next we will take these two mappings and for each we will send $t_2 \rightarrow {t_2}^2$ while keeping everything else fixed. This will give us two more automorphisms. At this point we have a total of four automorphisms. To complete the group M permutate our t's, thus sending $t_1 \rightarrow t_2$ and $t_2 \rightarrow t_1$, giving us four more automorphisms.

The first two automorphisms we found represent the group $U_1(3)$ which is of order 2. Then we found the next set of automorphisms that represent another group $U_2(3)$. We find the last four autmorphisms by permutating our group by S_2, doubling the four that we already had which completes our group $Aut(3^{*2})$, bringing the order of $Aut(3^{*2})$ to 8, $(2 \times 2 \times 2)$. We see that $Aut(3^{*2}) \cong U(3) \wr S_2$.

Sending $t_1 \to {t_1}^1$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	t_2	$\to$	t_2	3
4	${t_2}^2$	$\to$	${t_2}^2$	4

e

Sending $t_1 \to {t_1}^2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	t_1	1
3	t_2	$\to$	t_2	3
4	${t_2}^2$	$\to$	${t_2}^2$	4

(1,2)

Sending $t_2 \to {t_2}^2$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	t_2	$\to$	${t_2}^2$	4
4	${t_2}^2$	$\to$	t_2	3

(3,4)

Sending $t_2 \to {t_2}^2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	t_1	1
3	t_2	$\to$	${t_2}^2$	4
4	${t_2}^2$	$\to$	t_2	3

(1,2)(3,4)

$t_1 \to t_2$, $t_2 \to t_1$

1	t_1	$\to$	t_2	3
2	${t_1}^2$	$\to$	${t_2}^2$	4
3	t_2	$\to$	t_1	1
4	${t_2}^2$	$\to$	${t_1}^2$	2

(1,3)(2,4)

$t_1 \to t_2$, $t_2 \to t_1$

1	t_1	$\to$	${t_2}^2$	4
2	${t_1}^2$	$\to$	t_2	3
3	t_2	$\to$	t_1	1
4	${t_2}^2$	$\to$	${t_1}^2$	2

(1,4,2,3)

$t_1 \to t_2$, $t_2 \to t_1$

1	t_1	$\to$	t_2	3
2	${t_1}^2$	$\to$	${t_2}^2$	4
3	t_2	$\to$	${t_1}^2$	2
4	${t_2}^2$	$\to$	t_1	1

(1,3,2,4)

$t_1 \to t_2$, $t_2 \to t_1$

1	t_1	$\to$	${t_2}^2$	4
2	${t_1}^2$	$\to$	t_2	3
3	t_2	$\to$	${t_1}^2$	2
4	${t_2}^2$	$\to$	t_1	1

(1,4)(2,3)

5.4 Example $Aut(5^{*2})$

Here is another example of finding the group of all monomial automorphisms, M, of 5^{*2}. We know that 5^{*2} is on two letters, t_1 and t_2, each of which is of order 5. Now $U(5) = \{1, 2, 3, 4\}$ is a group under multiplication mod 5 and is the set of numbers that are coprime to 5 since $U(n) = \{a|1 \leq a \leq n, gcd(a, n) = 1\}$. We will now choose a t, say t_1 to send to ${t_1}^1 = t_1$ and fix all other t's. Similarly the second, third and fourth mappings send $t_1 \to {t_1}^2$, $t_1 \to {t_1}^3$, and $t_1 \to {t_1}^4$.

So we have four automorphisms. Next we will take these four mappings and for each we will send $t_2 \to {t_2}^2$, while keeping everything else fixed. We will repeat this process by sending $t_2 \to {t_2}^3$ and again for $t_2 \to {t_2}^4$. This will give us four times the number of automorphisms. At this point we have a total of sixteen automorphisms (4×4). To complete the group M we will take all sixteen mappings and will change all the t_1's on the right to t_2's and all t_2's will become t_1's. This will give us sixteen more automorphisms and will complete our group M.

The first two automorphisms we found represent the group $U(5)$ which is of order 4. Then we found the next set of automorphisms that represent another group $U(5)$. When we find the last set of autmorphisms we double the sixteen that we already had which completes our group M, bringing the order of M to 32, $(4 \times 4 \times 2)$. When we find the isomorphism type of M we see that it is the wreath product $U(5) \wr S_2$.

$t_1 \to t_1$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	${t_1}^3$	$\to$	${t_1}^3$	3
4	${t_1}^4$	$\to$	${t_1}^4$	4
5	t_2	$\to$	t_2	5
6	${t_2}^2$	$\to$	${t_2}^2$	6
7	${t_2}^3$	$\to$	${t_2}^3$	7
8	${t_2}^4$	$\to$	${t_2}^4$	8

e

$t_1 \to {t_1}^2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	${t_1}^4$	4
3	${t_1}^3$	$\to$	t_1	1
4	${t_1}^4$	$\to$	${t_1}^3$	3
5	t_2	$\to$	t_2	5
6	${t_2}^2$	$\to$	${t_2}^2$	6
7	${t_2}^3$	$\to$	${t_2}^3$	7
8	${t_2}^4$	$\to$	${t_2}^4$	8

(1,2,4,3)

$t_1 \to {t_1}^3$

1	t_1	$\to$	${t_1}^3$	3
2	${t_1}^2$	$\to$	t_1	1
3	${t_1}^3$	$\to$	${t_1}^4$	4
4	${t_1}^4$	$\to$	${t_1}^2$	2
5	t_2	$\to$	t_2	5
6	${t_2}^2$	$\to$	${t_2}^2$	6
7	${t_2}^3$	$\to$	${t_2}^3$	7
8	${t_2}^4$	$\to$	${t_2}^4$	8

(1,3,4,2)

$t_1 \to {t_1}^4$

1	t_1	$\to$	${t_1}^4$	4
2	${t_1}^2$	$\to$	${t_1}^3$	3
3	${t_1}^3$	$\to$	${t_1}^2$	2
4	${t_1}^4$	$\to$	t_1	1
5	t_2	$\to$	t_2	5
6	${t_2}^2$	$\to$	${t_2}^2$	6
7	${t_2}^3$	$\to$	${t_2}^3$	7
8	${t_2}^4$	$\to$	${t_2}^4$	8

(1,4)(2,3)

$t_2 \to {t_2}^2$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	${t_1}^3$	$\to$	${t_1}^3$	3
4	${t_1}^4$	$\to$	${t_1}^4$	4
5	t_2	$\to$	${t_2}^2$	6
6	${t_2}^2$	$\to$	${t_2}^4$	8
7	${t_2}^3$	$\to$	t_2	5
8	${t_2}^4$	$\to$	${t_2}^3$	7

(5,6,8,7)

$t_2 \to {t_2}^2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	${t_1}^4$	4
3	${t_1}^3$	$\to$	t_1	1
4	${t_1}^4$	$\to$	${t_1}^3$	3
5	t_2	$\to$	${t_2}^2$	6
6	${t_2}^2$	$\to$	${t_2}^4$	8
7	${t_2}^3$	$\to$	t_2	5
8	${t_2}^4$	$\to$	${t_2}^3$	7

(1,2,4,3)(5,6,8,7)

$t_2 \to {t_2}^2$

1	t_1	$\to$	${t_1}^3$	3
2	${t_1}^2$	$\to$	t_1	1
3	${t_1}^3$	$\to$	${t_1}^4$	4
4	${t_1}^4$	$\to$	${t_1}^2$	2
5	t_2	$\to$	${t_2}^2$	6
6	${t_2}^2$	$\to$	${t_2}^4$	8
7	${t_2}^3$	$\to$	t_2	5
8	${t_2}^4$	$\to$	${t_2}^3$	7

(1,3,4,2)(5,6,8,7)

$t_2 \to {t_2}^2$

1	t_1	$\to$	${t_1}^4$	4
2	${t_1}^2$	$\to$	${t_1}^2$	3
3	${t_1}^3$	$\to$	${t_1}^3$	2
4	${t_1}^4$	$\to$	t_1	1
5	t_2	$\to$	${t_2}^2$	6
6	${t_2}^2$	$\to$	${t_2}^4$	8
7	${t_2}^3$	$\to$	t_2	5
8	${t_2}^4$	$\to$	${t_2}^3$	7

(1,4)(2,3)(5,6,8,7)

$t_2 \to {t_2}^3$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	${t_1}^3$	$\to$	${t_1}^3$	3
4	${t_1}^4$	$\to$	${t_1}^4$	4
5	t_2	$\to$	${t_2}^3$	7
6	${t_2}^2$	$\to$	t_2	5
7	${t_2}^3$	$\to$	${t_2}^4$	8
8	${t_2}^4$	$\to$	${t_2}^2$	6

(5,7,8,6)

$t_2 \to {t_2}^3$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	${t_1}^4$	4
3	${t_1}^3$	$\to$	t_1	1
4	${t_1}^4$	$\to$	${t_1}^3$	3
5	t_2	$\to$	${t_2}^3$	7
6	${t_2}^2$	$\to$	t_2	5
7	${t_2}^3$	$\to$	${t_2}^4$	8
8	${t_2}^4$	$\to$	${t_2}^2$	6

(1,2,4,3)(5,7,8,6)

$t_2 \to {t_2}^3$

1	t_1	$\to$	${t_1}^3$	3
2	${t_1}^2$	$\to$	t_1	1
3	${t_1}^3$	$\to$	${t_1}^4$	4
4	${t_1}^4$	$\to$	${t_1}^2$	2
5	t_2	$\to$	${t_2}^3$	7
6	${t_2}^2$	$\to$	t_2	5
7	${t_2}^3$	$\to$	${t_2}^4$	8
8	${t_2}^4$	$\to$	${t_2}^2$	6

(1,3,4,2)(5,7,8,6)

$t_2 \to {t_2}^3$

1	t_1	$\to$	${t_1}^4$	4
2	${t_1}^2$	$\to$	${t_1}^3$	3
3	${t_1}^3$	$\to$	${t_1}^2$	2
4	${t_1}^4$	$\to$	t_1	1
5	t_2	$\to$	${t_2}^3$	7
6	${t_2}^2$	$\to$	t_2	5
7	${t_2}^3$	$\to$	${t_2}^4$	8
8	${t_2}^4$	$\to$	${t_2}^2$	6

(1,4)(2,3)(5,7,8,6)

$t_2 \to {t_2}^4$				
1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	${t_1}^3$	$\to$	${t_1}^3$	3
4	${t_1}^4$	$\to$	${t_1}^4$	4
5	t_2	$\to$	${t_2}^4$	8
6	${t_2}^2$	$\to$	${t_2}^3$	7
7	${t_2}^3$	$\to$	${t_2}^2$	6
8	${t_2}^4$	$\to$	t_2	5
(5,8)(6,7)				

$t_2 \to {t_2}^4$				
1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	${t_1}^4$	4
3	${t_1}^3$	$\to$	t_1	1
4	${t_1}^4$	$\to$	${t_1}^3$	3
5	t_2	$\to$	${t_2}^4$	8
6	${t_2}^2$	$\to$	${t_2}^3$	7
7	${t_2}^3$	$\to$	${t_2}^2$	6
8	${t_2}^4$	$\to$	t_2	5
(1,2,4,3)(5,8)(6,7)				

$t_2 \to {t_2}^4$				
1	t_1	$\to$	${t_1}^3$	3
2	${t_1}^2$	$\to$	t_1	1
3	${t_1}^3$	$\to$	${t_1}^4$	4
4	${t_1}^4$	$\to$	${t_1}^2$	2
5	t_2	$\to$	${t_2}^4$	8
6	${t_2}^2$	$\to$	${t_2}^3$	7
7	${t_2}^3$	$\to$	${t_2}^2$	6
8	${t_2}^4$	$\to$	t_2	5
(1,3,4,2)(5,8)(6,7)				

$t_2 \to {t_2}^4$				
1	t_1	$\to$	${t_1}^4$	4
2	${t_1}^2$	$\to$	${t_1}^3$	3
3	${t_1}^3$	$\to$	${t_1}^2$	2
4	${t_1}^4$	$\to$	t_1	1
5	t_2	$\to$	${t_2}^4$	8
6	${t_2}^2$	$\to$	${t_2}^3$	7
7	${t_2}^3$	$\to$	${t_2}^2$	6
8	${t_2}^4$	$\to$	t_2	5
(1,4)(2,3)(5,8)(6,7)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	t_2	5
2	${t_1}^2$	$\to$	${t_2}^2$	6
3	${t_1}^3$	$\to$	${t_2}^3$	7
4	${t_1}^4$	$\to$	${t_2}^4$	8
5	t_2	$\to$	t_1	1
6	${t_2}^2$	$\to$	${t_1}^2$	2
7	${t_2}^3$	$\to$	${t_1}^3$	3
8	${t_2}^4$	$\to$	${t_1}^4$	4
(1,5)(2,6)(3,7)(4,8)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^2$	6
2	${t_1}^2$	$\to$	${t_2}^4$	8
3	${t_1}^3$	$\to$	t_2	5
4	${t_1}^4$	$\to$	${t_2}^3$	7
5	t_2	$\to$	t_1	1
6	${t_2}^2$	$\to$	${t_1}^2$	2
7	${t_2}^3$	$\to$	${t_1}^3$	3
8	${t_2}^4$	$\to$	${t_1}^4$	4
(1,6,2,8,4,7,3,5)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^3$	7
2	${t_1}^2$	$\to$	t_2	5
3	${t_1}^3$	$\to$	${t_2}^4$	8
4	${t_1}^4$	$\to$	${t_2}^2$	6
5	t_2	$\to$	t_1	1
6	${t_2}^2$	$\to$	${t_1}^2$	2
7	${t_2}^3$	$\to$	${t_1}^3$	3
8	${t_2}^4$	$\to$	${t_1}^4$	4
(1,7,3,8,4,6,2,5)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^4$	8
2	${t_1}^2$	$\to$	${t_2}^3$	7
3	${t_1}^3$	$\to$	${t_2}^2$	6
4	${t_1}^4$	$\to$	t_2	5
5	t_2	$\to$	t_1	1
6	${t_2}^2$	$\to$	${t_1}^2$	2
7	${t_2}^3$	$\to$	${t_1}^3$	3
8	${t_2}^4$	$\to$	${t_1}^4$	4
(1,8,4,5)(2,7,3,6)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	t_2	5
2	${t_1}^2$	$\to$	${t_2}^2$	6
3	${t_1}^3$	$\to$	${t_2}^3$	7
4	${t_1}^4$	$\to$	${t_2}^4$	8
5	t_2	$\to$	${t_1}^2$	2
6	${t_2}^2$	$\to$	${t_1}^4$	4
7	${t_2}^3$	$\to$	t_1	1
8	${t_2}^4$	$\to$	${t_1}^3$	3
(1,5,2,6,4,8,3,7)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^2$	6
2	${t_1}^2$	$\to$	${t_2}^4$	8
3	${t_1}^3$	$\to$	t_2	5
4	${t_1}^4$	$\to$	${t_2}^3$	7
5	t_2	$\to$	${t_1}^2$	2
6	${t_2}^2$	$\to$	${t_1}^4$	4
7	${t_2}^3$	$\to$	t_1	1
8	${t_2}^4$	$\to$	${t_1}^3$	3
(1,6,4,7)(2,8,3,5)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^3$	7
2	${t_1}^2$	$\to$	t_2	5
3	${t_1}^3$	$\to$	${t_2}^4$	8
4	${t_1}^4$	$\to$	${t_2}^2$	6
5	t_2	$\to$	${t_1}^2$	2
6	${t_2}^2$	$\to$	${t_1}^4$	4
7	${t_2}^3$	$\to$	t_1	1
8	${t_2}^4$	$\to$	${t_1}^3$	3
(1,7)(2,5)(3,8)(4,6)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^4$	8
2	${t_1}^2$	$\to$	${t_2}^3$	7
3	${t_1}^3$	$\to$	${t_2}^2$	6
4	${t_1}^4$	$\to$	t_2	5
5	t_2	$\to$	${t_1}^2$	2
6	${t_2}^2$	$\to$	${t_1}^4$	4
7	${t_2}^3$	$\to$	t_1	1
8	${t_2}^4$	$\to$	${t_1}^3$	3
(1,8,3,6,4,5,2,7)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	t_2	5
2	${t_1}^2$	$\to$	${t_2}^2$	6
3	${t_1}^3$	$\to$	${t_2}^3$	7
4	${t_1}^4$	$\to$	${t_2}^4$	8
5	t_2	$\to$	${t_1}^3$	3
6	${t_2}^2$	$\to$	t_1	1
7	${t_2}^3$	$\to$	${t_1}^4$	4
8	${t_2}^4$	$\to$	${t_1}^2$	2
(1,5,3,7,4,8,2,6)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^2$	6
2	${t_1}^2$	$\to$	${t_2}^4$	8
3	${t_1}^3$	$\to$	t_2	5
4	${t_1}^4$	$\to$	${t_2}^3$	7
5	t_2	$\to$	${t_1}^3$	3
6	${t_2}^2$	$\to$	t_1	1
7	${t_2}^3$	$\to$	${t_1}^4$	4
8	${t_2}^4$	$\to$	${t_1}^2$	2
(1,6)(2,8)(3,5)(4,7)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^3$	7
2	${t_1}^2$	$\to$	t_2	5
3	${t_1}^3$	$\to$	${t_2}^4$	8
4	${t_1}^4$	$\to$	${t_2}^2$	6
5	t_2	$\to$	${t_1}^3$	3
6	${t_2}^2$	$\to$	t_1	1
7	${t_2}^3$	$\to$	${t_1}^4$	4
8	${t_2}^4$	$\to$	${t_1}^2$	2
(1,7,4,6,2,5,3,8)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^4$	8
2	${t_1}^2$	$\to$	${t_2}^3$	6
3	${t_1}^3$	$\to$	${t_2}^2$	7
4	${t_1}^4$	$\to$	t_2	5
5	t_2	$\to$	${t_1}^3$	3
6	${t_2}^2$	$\to$	t_1	1
7	${t_2}^3$	$\to$	${t_1}^4$	4
8	${t_2}^4$	$\to$	${t_1}^2$	2
(1,8,2,7,4,5,3,6)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	t_2	5
2	${t_1}^2$	$\to$	${t_2}^2$	6
3	${t_1}^3$	$\to$	${t_2}^3$	7
4	${t_1}^4$	$\to$	${t_2}^4$	8
5	t_2	$\to$	${t_1}^4$	4
6	${t_2}^2$	$\to$	${t_1}^3$	3
7	${t_2}^3$	$\to$	${t_1}^2$	2
8	${t_2}^4$	$\to$	t_1	1
(1,5,4,8)(2,6,3,7)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^2$	6
2	${t_1}^2$	$\to$	${t_2}^4$	8
3	${t_1}^3$	$\to$	t_2	5
4	${t_1}^4$	$\to$	${t_2}^3$	7
5	t_2	$\to$	${t_1}^4$	4
6	${t_2}^2$	$\to$	${t_1}^3$	3
7	${t_2}^3$	$\to$	${t_1}^2$	2
8	${t_2}^4$	$\to$	t_1	1
(1,6,3,5,4,7,2,8)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^3$	7
2	${t_1}^2$	$\to$	t_2	5
3	${t_1}^3$	$\to$	${t_2}^4$	8
4	${t_1}^4$	$\to$	${t_2}^2$	6
5	t_2	$\to$	${t_1}^4$	4
6	${t_2}^2$	$\to$	${t_1}^3$	3
7	${t_2}^3$	$\to$	${t_1}^2$	2
8	${t_2}^4$	$\to$	t_1	1
(1,7,2,5,4,6,3,8)				

$t_1 \to t_2$ and $t_2 \to t_1$				
1	t_1	$\to$	${t_2}^4$	8
2	${t_1}^2$	$\to$	${t_2}^3$	7
3	${t_1}^3$	$\to$	${t_2}^2$	6
4	${t_1}^4$	$\to$	t_2	5
5	t_2	$\to$	${t_1}^4$	4
6	${t_2}^2$	$\to$	${t_1}^3$	3
7	${t_2}^3$	$\to$	${t_1}^2$	2
8	${t_2}^4$	$\to$	t_1	1
(1,8)(2,7)(3,6)(4,5)				

5.5 Example $Aut(3^{*3})$

Here is another example of finding the group of all monomial automorphisms, M, of 3^{*3}. We know that 3^{*3} is on three letters, t_1, t_2 and t_3, each of which is of order 3. Now $U(3) = \{1, 2\}$ is a group under multiplication mod 3 and is the set of numbers that are coprime to 3 since $U(n) = \{a|1 \leq a \leq n, gcd(a, n) = 1\}$. We will now choose a t, say t_1 to send to ${t_1}^1 = t_1$ and fix all other t's. Similarly the second mapping sends $t_1 \to {t_1}^2$.

So we have two automorphisms. Next we will take these two mappings and for each we will send $t_2 \to {t_2}^2$ while keeping everything else fixed. This will give us two more automorphisms. At this point we have a total of four automorphisms. Then for the four automorphisms we send $t_3 \to {t_3}^2$, which brings us to eight automorphisms. To complete the group M we will take all eight mappings and will permutate $t_1 \to t_2$ and $t_2 \to t_1$, (1,2). Similarly we will also perform the following permutations,

$$t_1 \to t_3, \text{ (1,3)}$$
$$t_2 \to t_3, \text{ (2,3)}$$
$$t_1 \to t_2 \to t_3, \text{ (1,2,3)}$$
$$\text{and } t_1 \to t_3 \to t_2, \text{ (1,3,2).}$$

The first two automorphisms we found represent the group $U_1(3)$ which is of order 2. Then we found the next two sets of automorphisms that represent groups $U_2(3)$ and $U_3(3)$. When permutate the set of t's the set of permutations is S_3. Applying the permutations to our previous mappings of $U_1(3)$, $U_2(3)$, and $U_3(3)$ we find the last set of autmorphisms, multiplying the eight automorphisms by six, which completes our group M, bringing the order of M to 48, $(2 \times 2 \times 2 \times 6)$. When we find the isomorphism type of M we see that it is the wreath product $U(3) \wr S_3$.

Sending $t_1 \to {t_1}^1$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	t_2	$\to$	t_2	3
4	${t_2}^2$	$\to$	${t_2}^2$	4
5	t_3	$\to$	t_3	5
6	${t_3}^2$	$\to$	${t_3}^2$	6

e

Sending $t_1 \to {t_1}^2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	t_1	1
3	t_2	$\to$	t_2	3
4	${t_2}^2$	$\to$	${t_2}^2$	4
5	t_3	$\to$	t_3	5
6	${t_3}^2$	$\to$	${t_3}^2$	6

(1,2)

Sending $t_2 \to {t_2}^2$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	t_2	$\to$	${t_2}^2$	4
4	${t_2}^2$	$\to$	t_2	3
5	t_3	$\to$	t_3	5
6	${t_3}^2$	$\to$	${t_3}^2$	6

(3,4)

Sending $t_2 \to {t_2}^2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	t_1	1
3	t_2	$\to$	${t_2}^2$	4
4	${t_2}^2$	$\to$	t_2	3
5	t_3	$\to$	t_3	5
6	${t_3}^2$	$\to$	${t_3}^2$	6

(1,2)(3,4)

Sending $t_3 \to {t_3}^2$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	t_2	$\to$	t_2	3
4	${t_2}^2$	$\to$	${t_2}^2$	4
5	t_3	$\to$	${t_3}^2$	6
6	${t_3}^2$	$\to$	t_3	5

(5,6)

Sending $t_3 \to {t_3}^2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	t_1	1
3	t_2	$\to$	t_2	3
4	${t_2}^2$	$\to$	${t_2}^2$	4
5	t_3	$\to$	${t_3}^2$	6
6	${t_3}^2$	$\to$	t_3	5

(1,2)(5,6)

Sending $t_3 \to {t_3}^2$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	t_2	$\to$	${t_2}^2$	4
4	${t_2}^2$	$\to$	t_2	3
5	t_3	$\to$	${t_3}^2$	6
6	${t_3}^2$	$\to$	t_3	5

(3,4)(5,6)

Sending $t_3 \to {t_3}^2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	t_1	1
3	t_2	$\to$	${t_2}^2$	4
4	${t_2}^2$	$\to$	t_2	3
5	t_3	$\to$	${t_3}^2$	6
6	${t_3}^2$	$\to$	t_3	5

(1,2)(3,4)(5,6)

$t_1 \to t_2$, $t_2 \to t_1$

1	t_1	$\to$	t_2	3
2	${t_1}^2$	$\to$	${t_2}^2$	4
3	t_2	$\to$	t_1	1
4	${t_2}^2$	$\to$	${t_1}^2$	2
5	t_3	$\to$	t_3	5
6	${t_3}^2$	$\to$	${t_3}^2$	6

(1,3)(2,4)

$t_1 \to t_2$, $t_2 \to t_1$

1	t_1	$\to$	${t_2}^2$	4
2	${t_1}^2$	$\to$	t_2	3
3	t_2	$\to$	t_1	1
4	${t_2}^2$	$\to$	${t_1}^2$	2
5	t_3	$\to$	t_3	5
6	${t_3}^2$	$\to$	${t_3}^2$	6

(1,4,2,3)

$t_1 \to t_2,\ t_2 \to t_1$				
1	t_1	$\to$	t_2	3
2	${t_1}^2$	$\to$	${t_2}^2$	4
3	t_2	$\to$	${t_1}^2$	2
4	${t_2}^2$	$\to$	t_1	1
5	t_3	$\to$	t_3	5
6	${t_3}^2$	$\to$	${t_3}^2$	6

(1,3,2,4)

$t_1 \to t_2,\ t_2 \to t_1$				
1	t_1	$\to$	${t_2}^2$	4
2	${t_1}^2$	$\to$	t_2	3
3	t_2	$\to$	${t_1}^2$	2
4	${t_2}^2$	$\to$	t_1	1
5	t_3	$\to$	t_3	5
6	${t_3}^2$	$\to$	${t_3}^2$	6

(1,4)(2,3)

$t_1 \to t_2,\ t_2 \to t_1$				
1	t_1	$\to$	t_2	3
2	${t_1}^2$	$\to$	${t_2}^2$	4
3	t_2	$\to$	t_1	1
4	${t_2}^2$	$\to$	${t_1}^2$	2
5	t_3	$\to$	${t_3}^2$	6
6	${t_3}^2$	$\to$	t_3	5

(1,3)(2,4)(5,6)

$t_1 \to t_2,\ t_2 \to t_1$				
1	t_1	$\to$	${t_2}^2$	4
2	${t_1}^2$	$\to$	t_2	3
3	t_2	$\to$	t_1	1
4	${t_2}^2$	$\to$	${t_1}^2$	2
5	t_3	$\to$	${t_3}^2$	6
6	${t_3}^2$	$\to$	t_3	5

(1,4,2,3)(5,6)

$t_1 \to t_2,\ t_2 \to t_1$				
1	t_1	$\to$	t_2	3
2	${t_1}^2$	$\to$	${t_2}^2$	4
3	t_2	$\to$	${t_1}^2$	2
4	${t_2}^2$	$\to$	t_1	1
5	t_3	$\to$	${t_3}^2$	6
6	${t_3}^2$	$\to$	t_3	5

(1,3,2,4)(5,6)

$t_1 \to t_2,\ t_2 \to t_1$				
1	t_1	$\to$	${t_2}^2$	4
2	${t_1}^2$	$\to$	t_2	3
3	t_2	$\to$	${t_1}^2$	2
4	${t_2}^2$	$\to$	t_1	1
5	t_3	$\to$	${t_3}^2$	6
6	${t_3}^2$	$\to$	t_3	5

(1,4)(2,3)(5,6)

$t_1 \to t_3,\ t_3 \to t_1$				
1	t_1	$\to$	t_3	5
2	${t_1}^2$	$\to$	${t_3}^2$	6
3	t_2	$\to$	t_2	3
4	${t_2}^2$	$\to$	${t_2}^2$	4
5	t_3	$\to$	t_1	1
6	${t_3}^2$	$\to$	${t_1}^2$	2

(1,5)(2,6)

$t_1 \to t_3,\ t_3 \to t_1$				
1	t_1	$\to$	${t_3}^2$	6
2	${t_1}^2$	$\to$	t_3	5
3	t_2	$\to$	t_2	3
4	${t_2}^2$	$\to$	${t_2}^2$	4
5	t_3	$\to$	t_1	1
6	${t_3}^2$	$\to$	${t_1}^2$	2

(1,6,2,5)

$t_1 \to t_3,\ t_3 \to t_1$				
1	t_1	$\to$	t_3	5
2	${t_1}^2$	$\to$	${t_3}^2$	6
3	t_2	$\to$	${t_2}^2$	4
4	${t_2}^2$	$\to$	t_2	3
5	t_3	$\to$	t_1	1
6	${t_3}^2$	$\to$	${t_1}^2$	2

(1,5)(2,6)(3,4)

$t_1 \to t_3,\ t_3 \to t_1$				
1	t_1	$\to$	${t_3}^2$	6
2	${t_1}^2$	$\to$	t_3	5
3	t_2	$\to$	${t_2}^2$	4
4	${t_2}^2$	$\to$	t_2	3
5	t_3	$\to$	t_1	1
6	${t_3}^2$	$\to$	${t_1}^2$	2

(1,6,2,5)(3,4)

$t_1 \to t_3,\ t_3 \to t_1$

1	t_1	$\to$	t_3	5
2	${t_1}^2$	$\to$	${t_3}^2$	6
3	t_2	$\to$	t_2	3
4	${t_2}^2$	$\to$	${t_2}^2$	4
5	t_3	$\to$	${t_1}^2$	2
6	${t_3}^2$	$\to$	t_1	1

(1,5,2,6)

$t_1 \to t_3,\ t_3 \to t_1$

1	t_1	$\to$	${t_3}^2$	6
2	${t_1}^2$	$\to$	t_3	5
3	t_2	$\to$	t_2	3
4	${t_2}^2$	$\to$	${t_2}^2$	4
5	t_3	$\to$	${t_1}^2$	2
6	${t_3}^2$	$\to$	t_1	1

(1,6)(2,5)

$t_1 \to t_3,\ t_3 \to t_1$

1	t_1	$\to$	t_3	5
2	${t_1}^2$	$\to$	${t_3}^2$	6
3	t_2	$\to$	${t_2}^2$	4
4	${t_2}^2$	$\to$	t_2	3
5	t_3	$\to$	${t_1}^2$	2
6	${t_3}^2$	$\to$	t_1	1

(1,5,2,6)(3,4)

$t_1 \to t_3,\ t_3 \to t_1$

1	t_1	$\to$	${t_3}^2$	6
2	${t_1}^2$	$\to$	t_3	5
3	t_2	$\to$	${t_2}^2$	4
4	${t_2}^2$	$\to$	t_2	3
5	t_3	$\to$	${t_1}^2$	2
6	${t_3}^2$	$\to$	t_1	1

(1,6)(2,5)(3,4)

$t_2 \to t_3,\ t_3 \to t_2$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	t_2	$\to$	t_3	5
4	${t_2}^2$	$\to$	${t_3}^2$	6
5	t_3	$\to$	t_2	3
6	${t_3}^2$	$\to$	${t_2}^2$	4

(3,5)(4,6)

$t_2 \to t_3,\ t_3 \to t_2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	t_1	1
3	t_2	$\to$	t_3	5
4	${t_2}^2$	$\to$	${t_3}^2$	6
5	t_3	$\to$	t_2	3
6	${t_3}^2$	$\to$	${t_2}^2$	4

(1,2)(3,5)(4,6)

$t_2 \to t_3,\ t_3 \to t_2$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	t_2	$\to$	${t_3}^2$	6
4	${t_2}^2$	$\to$	t_3	5
5	t_3	$\to$	t_2	3
6	${t_3}^2$	$\to$	${t_2}^2$	4

(3,6,4,5)

$t_2 \to t_3,\ t_3 \to t_2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	t_1	1
3	t_2	$\to$	${t_3}^2$	6
4	${t_2}^2$	$\to$	t_3	5
5	t_3	$\to$	t_2	3
6	${t_3}^2$	$\to$	${t_2}^2$	4

(1,2)(3,6,4,5)

$t_2 \to t_3,\ t_3 \to t_2$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	t_2	$\to$	t_3	5
4	${t_2}^2$	$\to$	${t_3}^2$	6
5	t_3	$\to$	${t_2}^2$	4
6	${t_3}^2$	$\to$	t_2	3

(3,5,4,6)

$t_2 \to t_3,\ t_3 \to t_2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	t_1	1
3	t_2	$\to$	t_3	5
4	${t_2}^2$	$\to$	${t_3}^2$	6
5	t_3	$\to$	${t_2}^2$	4
6	${t_3}^2$	$\to$	t_2	3

(1,2)(3,5,4,6)

$t_2 \to t_3,\ t_3 \to t_2$

1	t_1	$\to$	t_1	1
2	${t_1}^2$	$\to$	${t_1}^2$	2
3	t_2	$\to$	${t_3}^2$	6
4	${t_2}^2$	$\to$	t_3	5
5	t_3	$\to$	${t_2}^2$	4
6	${t_3}^2$	$\to$	t_2	3

(3,6)(4,5)

$t_1 \to t_2 \to t_3$

1	t_1	$\to$	t_2	3
2	${t_1}^2$	$\to$	${t_2}^2$	4
3	t_2	$\to$	t_3	5
4	${t_2}^2$	$\to$	${t_3}^2$	6
5	t_3	$\to$	t_1	1
6	${t_3}^2$	$\to$	${t_1}^2$	2

(1,3,5)(2,4,6)

$t_1 \to t_2 \to t_3$

1	t_1	$\to$	t_2	3
2	${t_1}^2$	$\to$	${t_2}^2$	4
3	t_2	$\to$	${t_3}^2$	6
4	${t_2}^2$	$\to$	t_3	5
5	t_3	$\to$	t_1	1
6	${t_3}^2$	$\to$	${t_1}^2$	2

(1,3,6,2,4,5))

$t_1 \to t_2 \to t_3$

1	t_1	$\to$	t_2	3
2	${t_1}^2$	$\to$	${t_2}^2$	4
3	t_2	$\to$	t_3	5
4	${t_2}^2$	$\to$	${t_3}^2$	6
5	t_3	$\to$	${t_1}^2$	2
6	${t_3}^2$	$\to$	t_1	1

(1,3,5,2,4,6)

$t_1 \to t_2 \to t_3$

1	t_1	$\to$	t_2	3
2	${t_1}^2$	$\to$	${t_2}^2$	4
3	t_2	$\to$	${t_3}^2$	6
4	${t_2}^2$	$\to$	t_3	5
5	t_3	$\to$	${t_1}^2$	2
6	${t_3}^2$	$\to$	t_1	1

(1,3,6)(2,4,5)

$t_2 \to t_3,\ t_3 \to t_2$

1	t_1	$\to$	${t_1}^2$	2
2	${t_1}^2$	$\to$	t_1	1
3	t_2	$\to$	${t_3}^2$	6
4	${t_2}^2$	$\to$	t_3	5
5	t_3	$\to$	${t_2}^2$	4
6	${t_3}^2$	$\to$	t_2	3

(1,2)(3,6)(4,5)

$t_1 \to t_2 \to t_3$

1	t_1	$\to$	${t_2}^2$	4
2	${t_1}^2$	$\to$	t_2	3
3	t_2	$\to$	t_3	5
4	${t_2}^2$	$\to$	${t_3}^2$	6
5	t_3	$\to$	t_1	1
6	${t_3}^2$	$\to$	${t_1}^2$	2

(1,4,6,2,3,5)

$t_1 \to t_2 \to t_3$

1	t_1	$\to$	${t_2}^2$	4
2	${t_1}^2$	$\to$	t_2	3
3	t_2	$\to$	${t_3}^2$	6
4	${t_2}^2$	$\to$	t_3	5
5	t_3	$\to$	t_1	1
6	${t_3}^2$	$\to$	${t_1}^2$	2

(1,4,5))(2,3,6)

$t_1 \to t_2 \to t_3$

1	t_1	$\to$	${t_2}^2$	4
2	${t_1}^2$	$\to$	t_2	3
3	t_2	$\to$	t_3	5
4	${t_2}^2$	$\to$	${t_3}^2$	6
5	t_3	$\to$	${t_1}^2$	2
6	${t_3}^2$	$\to$	t_1	1

(1,4,6)(2,3,5)

$t_1 \to t_2 \to t_3$

1	t_1	$\to$	${t_2}^2$	4
2	${t_1}^2$	$\to$	t_2	3
3	t_2	$\to$	${t_3}^2$	6
4	${t_2}^2$	$\to$	t_3	5
5	t_3	$\to$	${t_1}^2$	2
6	${t_3}^2$	$\to$	t_1	1

(1,4,5,2,3,6)

	$t_1 \to t_3 \to t_2$			
1	t_1	$\to$	t_3	5
2	${t_1}^2$	$\to$	${t_3}^2$	6
3	t_2	$\to$	t_1	1
4	${t_2}^2$	$\to$	${t_1}^2$	2
5	t_3	$\to$	t_2	3
6	${t_3}^2$	$\to$	${t_2}^2$	4
	(1,5,3)(2,6,4)			

	$t_1 \to t_3 \to t_2$			
1	t_1	$\to$	${t_3}^2$	6
2	${t_1}^2$	$\to$	t_3	5
3	t_2	$\to$	t_1	1
4	${t_2}^2$	$\to$	${t_1}^2$	2
5	t_3	$\to$	t_2	3
6	${t_3}^2$	$\to$	${t_2}^2$	4
	(1,6,4,2,5,3)			

	$t_1 \to t_3 \to t_2$			
1	t_1	$\to$	t_3	5
2	${t_1}^2$	$\to$	${t_3}^2$	6
3	t_2	$\to$	${t_1}^2$	2
4	${t_2}^2$	$\to$	t_1	1
5	t_3	$\to$	t_2	3
6	${t_3}^2$	$\to$	${t_2}^2$	4
	(1,5,3,2,6,4)			

	$t_1 \to t_3 \to t_2$			
1	t_1	$\to$	${t_3}^2$	6
2	${t_1}^2$	$\to$	t_3	5
3	t_2	$\to$	${t_1}^2$	2
4	${t_2}^2$	$\to$	t_1	1
5	t_3	$\to$	t_2	3
6	${t_3}^2$	$\to$	${t_2}^2$	4
	(1,6,4)(2,5,3)			

	$t_1 \to t_3 \to t_2$			
1	t_1	$\to$	t_3	5
2	${t_1}^2$	$\to$	${t_3}^2$	6
3	t_2	$\to$	t_1	1
4	${t_2}^2$	$\to$	${t_1}^2$	2
5	t_3	$\to$	${t_2}^2$	4
6	${t_3}^2$	$\to$	t_2	3
	(1,5,4,2,6,3)			

	$t_1 \to t_3 \to t_2$			
1	t_1	$\to$	${t_3}^2$	6
2	${t_1}^2$	$\to$	t_3	5
3	t_2	$\to$	t_1	1
4	${t_2}^2$	$\to$	${t_1}^2$	2
5	t_3	$\to$	${t_2}^2$	4
6	${t_3}^2$	$\to$	t_2	3
	(1,6,3)(2,5,4)			

	$t_1 \to t_3 \to t_2$			
1	t_1	$\to$	t_3	5
2	${t_1}^2$	$\to$	${t_3}^2$	6
3	t_2	$\to$	${t_1}^2$	2
4	${t_2}^2$	$\to$	t_1	1
5	t_3	$\to$	${t_2}^2$	4
6	${t_3}^2$	$\to$	t_2	3
	(1,5,4)(2,6,3)			

	$t_1 \to t_3 \to t_2$			
1	t_1	$\to$	${t_3}^2$	6
2	${t_1}^2$	$\to$	t_3	5
3	t_2	$\to$	${t_1}^2$	2
4	${t_2}^2$	$\to$	t_1	1
5	t_3	$\to$	${t_2}^2$	4
6	${t_3}^2$	$\to$	t_2	3
	(1,6,3,2,5,4)			

Chapter 6

More Monomial Progenitors

6.1 Preliminiaries

Definition 6.1 (Representation of G)**.** *Let G be a finite group. A representation of G is a homomorphism $\phi : G \to GL(n, \mathbb{F})$, where $GL(n, \mathbb{F})$ is the group of $n \times n$ invertible matrices over a finite field F.*

Definition 6.2 (Character)**.** *The character χ afforded by the representation $\rho : G \to GL(n, \mathbb{F})$ is a function,*

$$\chi : G \to \mathbb{F}, \text{ given by } \chi(g) = Tr(g\rho)\ \forall g \in G.$$

Definition 6.3 (Character Table)**.** *Consider the character table as a the matrix $\left(\chi_\alpha^{(i)}\right)$. In a character table, let χ be the row vector $\left(h_\alpha \chi_\alpha^{(i)}\right)$.*

1. *Let $\overline{Y}$ be the conjugate of the row vector $\left(\chi_\alpha^{(i)}\right)$. If $i \neq j$, then the ordinary dot product $X \cdot \overline{Y} = 0$.*
2. *Then the ordinary dot product $X \cdot \left(\chi_\alpha^{(i)}\right) = |G|$.*

Definition 6.4 (Monomial Matrix)**.** *A monomial matrix is an $n \times n$ matrix that has exactly one non-zero entry in each row and column.*

Definition 6.5 (Monomial Representation of G)**.** *For a group G, a monomial representation is a representation $A : G \to GL(n, \mho)$, provided that $A(x)$ and $A(y)$ are monomial matrices.*

Definition 6.6 (Monomial Character of G)**.** *If a character ϕ of G is induced by a linear character of a subgroup H of G then ϕ is a monomial character.*

Definition 6.7 (Induced Character)**.** *Let $\Phi(u)$ be a character of H and $\phi(x) = 0$ if $x \notin H$. Then*

$$\phi^G(x) = \begin{cases} \phi(x) & if x \in H \\ 0 & if x \notin H \end{cases}$$

is an induced character of G.

6.2 Monomial Progenitor $3^{*3} : S_4$

We will first explain how we write a monomial progenitor using an example, $3^{*3} : S_4$, and then we will give images of a monomial progenitor that we have written, $3^{*3} :_m L_2(3)$.

As an example we will write a monomial progenitor $3^{*3} :_m S_4$ using MAGMA. We first begin by noting a presentation for S_4 is

$$< x, y | x^4, y^2, (x * y)^3 >.$$

We will induce a linear character of the subgroup H of G. We may choose any subgroup of S_4, whose index is equal to the degree of an irreducible character of G. To find a suitable subgroup we use the following code in MAGMA.

```
> Cr:=CharacterTable(G);
> for i in [1..#Cr] do i,Cr[i][1];
for> end for;
1 1
2 1
3 2
4 3
5 3
> S:=Subgroups(G);
> #S;
11
> for i in [1..#S] do i, Index(G,S[i]`subgroup);
for> end for;
1 24
2 12
```

```
3 12
4 8
5 6
6 6
7 6
8 4
9 3
10 2
11 1
```

From above we see that S[9] has index 3 and the degrees of Cr[4] and Cr[5] are also 3. This means that we will induce a linear character of $H = S[9]$ of G and our index of $A(x)$ and $A(y)$ will also be 3, that is, $A(x)$ and $A(y)$ will be represented by 3×3 matrices. The Character Tables of G and H are give below.

Table 6.1: Character Table of A_4

χ	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	2	2	0	-1	0
χ_4	3	-1	-1	0	1
χ_5	3	-1	1	0	-1

Table 6.2: Character Table of H

χ	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

We induce from the second character, χ_2. We can now find $A(x)$ and $A(y)$.

$$A(x) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$A(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Each column represents t_1, t_2, t_3 respectively. The elements of S_4 act as automorphisms of $< t_1 > * < t_2 > * < t_3 >$. Therefore, $a_{ij} = 1$ if the automorphism takes $t_i \to t_j$ and $a_{ij} = n$ if the automorphism takes $t_i \to t_j^n$. By translating the matrices of $A(x)$ and $A(y)$, we will be able to represent x and y as permutations.

Table 6.3: Automorphisms of $A(x)$

1	2	3	4	5	6
t_1	t_2	t_3	t_1^{-1}	t_2^{-1}	t_3^{-1}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2	t_1^{-1}	t_3^{-1}	t_2^{-1}	t_1	t_3
2	4	6	5	1	3

Table 6.4: Automorphisms of $A(y)$

1	2	3	4	5	6
t_1	t_2	t_3	t_1^{-1}	t_2^{-1}	t_3^{-1}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1	t_3	t_2	t_1^{-1}	t_3^{-1}	t_2^{-1}
1	3	2	4	6	5

From the tables above we get the following permutations for x and y.

$$x = (1, 2, 4, 5)(3, 6)$$
$$y = (2, 3)(5, 6)$$

Now we will label our $t \sim t_1$. We need to find the permutations of S_4 that normalize $< t_1 >$. The normalizer of $< t_1 >$ in S_4 is generated by 3 elements, (2,3)(5,6), (2,6)(3,5), (1,4)(2,5). Using the Schreier System we find that

$$(2, 3)(5, 6) = y$$
$$(2, 6)(3, 5) = x^2yx^2$$
$$(1, 4)(2, 5) = x^2.$$

The first two elements fix t_1, which means we will add them into the progenitor by writing them as the following.

$$(t, y)$$
$$(t, x^2yx^2)$$

The last element sends $1 \to 4$, therefore mapping $t_1 \to t_1^{-1}$. So we write it as

$$t^{x^2} = t^{-1}.$$

Now we can write a presentation for the monomial progenitor.

$$3^{*3} :_m S_4 =< x, y, t | x^4, y^2, (xy)^3, t^3, (t, y), (t, x^2yx^2), t^{x^2} = t^{-1} >$$

6.3 Monomial Progenitor $3^{*3} :_m L_2(3)$

We will write a monomial progenitor $3^{*3} :_m L_2(3)$ using MAGMA. We first begin by noting a presentation for $L_2(3)$ is

$$< x, y | x^3, y^2, (x * y)^3 >.$$

We will induce a linear character of the subgroup H of G. We may choose any subgroup of $L_2(3)$, whose index is equal to the degree of an irreducible character of G. To find a suitable subgroup we use to following code in MAGMA.

```
> Cr:=CharacterTable(G);
> for i in [1..#Cr] do i,Cr[i][1];
for> end for;
1 1
2 1
3 1
4 3
> S:=Subgroups(G);
> #S;
5
> for i in [1..#S] do i, Index(G,S[i]`subgroup);
for> end for;
1 12
2 6
3 4
4 3
5 1
```

From above we see that S[4] has index 3 and the degrees of Cr[4] is also 3. This means that we will induce a linear character of $H = S[4]$ of G and our index of $A(x)$ and $A(y)$ will also be 3, that is, $A(x)$ and $A(y)$ will be represented by 3×3 matrices.The Character Tables of G and H are given below.

Table 6.5: Character Table of $L_2(3)$

χ	C_1	C_2	C_3	C_4
χ_1	1	1	1	1
χ_2	1	1	J	-1-J
χ_3	1	1	-1-J	J
χ_4	3	-1	0	0
J is root of unity 3				

Table 6.6: Character Table of H

χ	C_1	C_2	C_3	C_4
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1

We induce from the second character, χ_2. We can now find $A(x)$ and $A(y)$.

$$A(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Each column represents t_1, t_2, t_3 respectively. The elements of S_4 act as automorphisms of $< t_1 > * < t_2 > * < t_3 >$. Therefore, $a_{ij} = 1$ if the automorphism takes $t_i \to t_j$ and $a_{ij} = n$ if the automorphism takes $t_i \to t_j^n$. By translating the matrices of $A(x)$ and $A(y)$, we will be able to represent x and y as permutations.

From the tables above we get the following permutations for x and y.

$$x = (1, 2, 3)(4, 5, 6)$$
$$y = (2, 5)(3, 6)$$

Table 6.7: Automorphisms of $A(x)$

1	2	3	4	5	6
t_1	t_2	t_3	t_1^{-1}	t_2^{-1}	t_3^{-1}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2	t_3	t_1	t_2^{-1}	t_3^{-1}	t_1^{-1}
2	3	1	5	6	4

Table 6.8: Automorphisms of $A(y)$

1	2	3	4	5	6
t_1	t_2	t_3	t_1^{-1}	t_2^{-1}	t_3^{-1}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1	t_2^{-1}	t_3^{-1}	t_1^{-1}	t_2	t_3
1	5	6	4	2	3

Now we will label our $t \sim t_1$. We need to find the permutations of S_4 that normalize $< t_1 >$. The normalizer of $< t_1 >$ in S_4 is generated by 2 elements, (2,5)(3,6) and (1,4)(2,5). Using the Schreier System we find that

$$(2,5)(3,6) = y$$
$$(1,4)(2,5) = xyx^{-1}.$$

The first element fixes t_1, which means we will add it into the progenitor by writing it as the following.

$$(t, y)$$

The second element sends $1 \rightarrow 4$, therefore mapping $t_1 \rightarrow t_1^{-1}$. So we write is as

$$t^{xyx^{-1}} = t^{-1}.$$

Now we can write a presentation for the monomial progenitor.

$$3^{*3} :_m L_2(3) =< x, y, t | x^3, y^2, (xy)^3, t^3, (t, y), t^{xyx^{-1}} = t^{-1} >$$

By adding the following relations to the progenitor we get the images given below.

$$(y * (t^y)^x * t * t^x)^a, (t^y * x * y * t)^b, (x * t * t^x)^c, (t * t^x * t^y * x)^d, (x * t)^e$$

Table 6.9: Homomorphic Images of $3^{*3} :_m L_2(3)$

a	b	c	d	e	Index	#G	Isomorphic Type
0	0	0	0	4	8	24	S_4
0	0	0	0	5	20	60	A_5
0	0	0	3	0	480	5760	$2^{\bullet}[(2^2 : 3) \times A_5]$
0	0	0	3	15	240	2880	$(2^2 : 3 \times A_5$
0	0	0	4	7	56	168	$L_2(7)$
0	0	0	4	9	816	2448	$L_2(17)$
0	0	0	4	10	1320	3960	$3 : PGL_2(11)$
0	0	0	4	12	5280	15840	$L_2(11) : S_4$
0	0	0	4	13	82160	246480	$L_2(79)$
0	0	0	4	14	366912	1100736	$L_2(7) \times [(3 \times L_2(13)) : 2]$
0	0	0	5	8	4960	14880	$L_2(31)$
0	0	0	7	7	364	1092	$L_2(13)$
0	0	0	8	7	3584	10752	$2^8 : L_2(7)$

Chapter 7

Transitive Progenitors

When writing progenitors of the form $m^{*n} : N$, we require that N be transitive. MAGMA allows us to find proper transitive groups on n letters by using the command `TransitiveGroups(n,i)`. Then we find the presentation for the group by using the command `SmallGroupDatabase()` which allows us to write the progenitor.

7.1 Transitive Group on 8 Letters

We will write a progenitor from a transitive group on 8 letters using MAGMA. There exist 50 transitive groups on 8 letters. The transitive group we chose was

```
N:=TransitiveGroup(8,19);
```

which has order 32. We can identify and find the presentation of N using the following commands

```
IdentifyGroup(N);
D:=SmallGroupDatabase ();
G:=SmallGroup(D,32,6);
```

where G is the new name in MAGMA for N. The command

```
FPGroup(G);
```

gives us a presentation of the group that allows us to form the progenitor.

We find that G has the following presentation with 5 generators.

$$G =< a, b, c, d, e | a^2 = d, b^2, c^2, d^2, e^2, b^a = b * c, c^a = c * e, c^b = c, d^a = d, d^b = d * e,$$
$$d^c = d, e^a = e, e^b = e, e^c = e, e^d = e >$$

This is the presentation of N for the progenitor $2^{*8} : N$. Our presentation is on 32 letters because the order of N is 32 but this can be written on 8 letters. We need to find the subgroups of G in order to find a faithful permutation representation of N on 8 letters.

```
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
#G1;
32
```

```
SL := Subgroups(G1);
T := {X`subgroup: X in SL};
#T;
```

T stores all the subgroups of G, and we will call each subgroup H. The command Core(G1,H) tells us the kernel of the subgroups, and we are interested in subgroups with kernel equal to 1. So we then run the following command.

```
TrivCore := {H:H in T| #Core(G1,H) eq 1};
```

From this we see that we have 9 subgroups whose kernel equals 1. Of those subgroups we now want the subgroup of smallest degree, 8.

```
Good := {H: H in TrivCore| Index(G1,H) eq mdeg};
```

Now we choose a representative from those subgroups of smallest degree.

```
H := Rep(Good);
```

And so now we can see that G1 is on 8 letters.

```
f,G1,K := CosetAction(G1,H);
G1;
Permutation group G1 acting on a set of cardinality 8
Order = 32 = 2^5
    (2, 5, 6, 8)(3, 7)
    (1, 2)(3, 5)(4, 6)(7, 8)
    (1, 3)(2, 5)(4, 7)(6, 8)
    (2, 6)(5, 8)
    (1, 4)(2, 6)(3, 7)(5, 8)
```

Now we introduce a new variable, t and let t t_1 and we find the stabilizer to complete the progenitor. We find that a and d commute with t. Our complete progenitor is the following.

$$2^8 : N =< a, b, c, d, e, t|a^2 = d,$$
$$b^2, c^2, d^2, e^2, b^a = b * c, c^a = c * e, c^b = c, d^a = d, d^b = d * e,$$
$$d^c = d, e^a = e, e^b = e, e^c = e, e^d = e, t^2, (t, a), (t, d) >$$

By adding the following relations to the progenitor we get the images given below.

$$(c * t^b)^i, (b * e * t * t^b)^j, (c * e * t * t^b)^k, (b * c * e * t^{b^2} * t^c * t^e)^l, (b * c * e * t^{b^2} * t^c * t)^m$$

7.2 Transitive Progenitor $2^{*8} : [2^{\bullet}(2^{2\bullet}2^2)]$

Table 7.1: Progenitors of $2^{*8} : 2^{\bullet}(2^{2\bullet}2^2)$, where $Kernel > 1$

i	j	k	l	m	Index	Order of G	Kernel
0	0	2	0	2	18	144	4
0	0	2	0	6	162	1296	4
0	0	2	0	10	450	3600	4
0	0	3	2	0	60	480	2
0	2	3	0	3	6552	52416	2
3	0	3	2	0	30	240	2
3	0	4	2	0	288	2304	2
4	2	0	0	5	40	320	2
4	2	3	6	6	48	384	2
4	2	4	0	10	720	5760	4
4	2	6	0	6	384	3072	4
4	2	6	7	0	1260	10080	2
5	2	2	0	0	25	200	2
5	2	4	0	6	92160	737280	2
5	2	5	0	5	165	1320	2
6	0	2	0	0	36	288	4
7	0	2	0	0	49	392	2
9	0	2	0	0	81	648	2
10	2	0	4	0	100	800	4
10	4	2	0	10	50	400	4

Table 7.2: Homomorphic Images of $2^{*8} : 2^{\bullet}(2^{2\bullet}2^2)$

i	j	k	l	m	Index	Order of G	Isomorphic Type
0	0	2	0	4	144	4608	$2^4 : [2^2 \times 3^2 : 2^2 : 2]$
0	0	2	0	8	576	18432	$2^{2\bullet}[2^4 : (2^2 \times 3^2 : 2^2 : 2)]$
0	0	3	0	2	960	30720	$2^8 : A_5 : 2$
3	0	3	0	0	240	7680	$2^6 : (A_5 : 2)$
4	0	0	6	3	12	384	$2^5 : (S_3 \times 2)$
4	6	0	0	4	2592	82944	$(2^3 \times 3^4) : (2^3 : 2^2)$

7.3 Transitive Progenitor $2^{*10} : [2^4 : 5]$

Here is another transitive progenitor on 10 letters factored by relations to give us the following list of the isomorphic types.

$$2^{*10} : [2^4 : 5] =< a, b, c, d, e, t | a^5, b^2, c^2, d^2, e^2, b^a = e, c^a = b * e, c^b = c, d^a = b * c * e,$$
$$d^b = d, d^c = d, e^a = b * c * d * e, e^b = e, e^c = e, e^d = e, t^2, (t, c), (t, d), (t, b * c * e),$$
$$(a * t^b)^i, (a^2 * e * t * t^b)^j, (a * e * t * t^b)^k, (b * a * e * t^(b^2) * t^a * t^e)^l, (b * a^2 * e * t^(a^2) * t^a * t)^m >$$

Table 7.3: Homomorphic Images of $2^{*10} : [2^4 : 5]$

i	j	k	l	m	Index	#G	Isomorphic Type
0	0	0	2	0	240	1200	$(2 \times 5) : (2 \times A_5)$
0	0	0	2	6	120	600	$D_5 \times A_5$
0	0	0	3	4	2976	14880	$L_2(31)$
4	0	5	4	8	32	160	$2^4 : D_5$
4	0	5	5	0	72	360	A_6
5	0	0	3	0	132	660	$L_2(11)$
5	0	0	4	6	6144	30720	$(2^5 : 2^4) : A_5$
7	0	0	3	7	504	2520	A_7

7.4 Transititve Progenitor $2^{*7} : L_2(7)$

Here is another transitive progenitor on 7 letters factored by relations to give us the following list of the isomorphic types.

$$2^{*7} : L_2(7) =< a, b, t | a^2, b^3, (a*b)^7, (b*a*b^(-1)*a)^4, t^2, (t, b), (t, a*b*a*b^-1*a), (a*b^-1* a*t)^n, (a*b^-1*a*t^a)^o, (a*b^-1*a*t^(a^(b^2)))^p, (a*b^-1*a*b*t)^q, (a*b^-1*a*b*t^(a^(b^2)))^r >$$

Table 7.4: Homomorphic Images of $2^{*7} : L_2(7)$

n	o	p	q	r	Index	Order of G	Isomorphic Type
0	0	0	0	4	128	21504	$2 \times [2^6 : L_2(7)$
0	0	0	4	0	16	2688	$2 \times [2^2 : L_2(7)]$
0	0	0	6	0	72	12096	$U_3(3) : 2$
0	0	6	0	8	8192	1376256	$2^7 : 2^6 : L_2(7)$
0	0	6	8	0	1024	172032	$2^7 : 2^3 : L_2(7)$
3	0	0	0	0	8	1344	$2^3 : L_2(7)$

Chapter 8

More Progenitors

Here are several more progenitors that I added relations to and investigated further over the course of our research.

8.1 Progenitor $2^{*10} : D_{20}$

$$2^{*10} : (D_{20}) =< x, y, t | x^1 0, y^2, (x * y)^2, t^2, (t, y), (x * t)^a, (x * y * t^x)^b,$$
$$(x^2 * y * t * t^x)^c, (t * t^x * t^{x^3})^d, (x * t^x)^e, (y * t)^f >$$

Table 8.1: Homomorphic Images of $2^{*10} : D_{20}$

a	b	c	d	e	f	Index	Order of G	Isomorphic Type
0	0	0	3	3	0	12	120	2×5
0	0	0	4	3	0	396	7920	$2^{\bullet}[3 : PGL_2(11)]$
0	0	2	0	0	0	20	400	$2^{\bullet}[2 \times (5^2 : 2 : 2)]$
0	0	3	0	3	0	18	360	$S_3 \times A_5$
0	0	3	3	0	0	192	3840	$2^5 : A_5$
0	0	4	0	3	0	132	2640	$2 \times PGL_2(11)$
0	0	5	0	3	0	66	1320	$PGL_2(11)$
0	0	5	2	5	0	5	10	D_5
0	0	6	2	5	0	81	1620	$3^4 : D_5 : 2$
0	0	10	2	5	0	625	12500	$5^4 : [2 \times D_5]$
0	0	10	5	3	0	342	6840	$2 \times L_2(19)$
0	3	0	6	6	0	576	11520	$(2^5 \times 3) : (2 \times A_5)$
0	4	3	6	8	0	144	2880	$(2 \times A_6) : 2^2$

8.2 Progenitor $2^{*11} : [D_{11}]$

$$2^{*11} : [11 : 2] =< a, b, t | b^2, (a^(-1) * b)^2, a^(-11), t^2, (t, b),$$
$$(a * t)^i, (a^2 * t)^j, (a^3 * t)^k, (a * t^(a^3) * a^4)^l >$$

Table 8.2: Homomorphic Images of $2^{*11} : [D_{11}]$

i	j	k	l	Index	Order of G	Isomorphic Type
0	3	0	4	552	12144	$PGL_2(23)$
0	3	5	5	60	1320	$PGL_2(11)$
0	3	6	6	8640	190080	$M12 : 2$

8.3 Progenitor $3^{*11} : [D_{11}]$

$$3^{*11} : [11 : 2] =< a, b, t | b^2, (a^(-1) * b)^2, a^(-11), t^3, (t, b),$$
$$(a * t)^i, (a^2 * t)^j, (a^3 * t)^k, (a * t^(a^3) * a^4)^l >$$

Table 8.3: Homomorphic Images of $3^{*11} : [D_{11}]$

i	j	k	l	Index	Order of G	Isomorphic Type
0	0	0	3	33	726	$11^2 : [2 \times 3]$

8.4 Progenitor $2^{*4} : [2 : 2^2]$

$$2^{*4} : [2 : 2^2] =< a, b, c, t | a^2, b^2, c^2, (a * b)^2, (b * c)^2, c * b * a * c * a, t^2, (t, c),$$
$$(b * t)^e, (c * t^a)^g, (c * a * t)^h, (t^b * c * a)^i >$$

8.5 Progenitor $3^{*4} : [2 : 2^2]$

$$3^{*4} : [2 : 2^2] =< a, b, c, t | a^2, b^2, c^2, (a * b)^2, (b * c)^2, c * b * a * c * a, t^3, (t, c),$$
$$(b * t)^e, (c * t^a)^g, (c * a * t)^h, (t^b * c * a)^i >$$

Table 8.4: Homomorphic Images of $2^{*4} : [2 : 2^2]$

e	g	h	i	Index	Order of G	Isomorphic Type
0	0	0	3	6	48	$2 \times S_4$
0	2	0	3	3	12	$2 \times S_3$
0	2	0	5	5	20	$2 \times D_5$
0	2	0	7	7	28	$2 \times (7 : 2)$
0	2	0	9	9	36	$2 \times (3 : S_3)$
0	3	0	5	15	120	$A_5 : 2$
0	3	0	6	27	216	$[3 : 3^2] : [2 : 2^2]$
0	3	0	8	168	1344	$2^{\bullet}[2 \times PGL_2(7)]$
0	3	0	9	855	6840	$PGL_2(19)$
0	4	0	5	40	320	$2^5 \bullet D_5$
0	5	0	5	165	1320	$PGL_2(11)$
0	6	0	4	36	288	$(2^2 \times 3^2) : (2^{\bullet}2^2)$
0	10	0	4	100	800	$(2^2 \times 5^2) : (2^{\bullet} : 2^2)$
3	0	0	4	9	72	$3^2 : (2^{\bullet} : 2^2)$
3	0	0	6	30	240	$2 \times A_5 : 2$
3	0	0	7	42	336	$PGL_2(7)$
3	0	0	9	612	4896	$PGL_2(17)$
5	0	0	4	25	200	$5^2 : 2^{\bullet}2^2$
5	0	0	5	90	720	$A_6 : 2$
7	0	0	4	49	392	$7^2 : 2^{\bullet}2^2$

Table 8.5: Homomorphic Images of $3^{*4} : [2 : 2^2]$

e	g	h	i	Index	Order of G	Isomorphic Type
0	0	0	3	12	96	$2^3 : [2^2 : 3]$
0	2	0	4	9	72	$3^2 : 2^{\bullet}2^2$
0	2	0	6	30	240	$[2 \times A_5] : 2$
0	2	0	7	42	336	$PGL_2(7)$
0	2	0	9	612	4896	$PGL_2(17)$
0	3	0	2	3	6	S_3
0	3	0	4	12	24	$2^2 : S_3$
0	3	0	5	120	960	$2^4 : A_5$
3	0	0	5	480	3840	$2^6 : A_5$
3	3	0	5	30	60	A_5
5	0	0	4	3600	28800	$2^2 : [(A_5 \times A_5) : 2]$

8.6 Progenitor $2^{*8} : [2 \times D_4]$

$2^{*8} : [2xD_4] =< a, b, c, d, t | a^2, b^2, c^4, d^2, (a*b)^2, b*c^{-1}*b*c, c^{-2}*b*d, (a*d)^2, c^{-1}*b*a*c^{-1}*a, t^2, (t, a*b), (c*t*t^c)^e, (b*t)^f, (a*t*t^b*t^c)^n, (a*b*t)^o, (d*t*t^{(b*c)})^p, (b*c*d*t*t^c)^q >$

Table 8.6: Homomorphic Images of $2^{*8} : 2 \times D_4$

e	f	n	o	p	q	Index	Order of G	Isomorphic Types
0	0	1	0	0	0	5	80	$2^2 : [D_5 : 2]$
0	0	3	0	2	0	3600	57600	$2^3 : [(A_5 \times A_5) : 2]$
0	0	3	0	3	2	3420	27360	$2^{\bullet}[2 \times PGL_2(19)]$
0	2	3	0	4	0	168	2688	$2^{\bullet}[2 \times PGL_2(7)]$
0	3	5	0	0	2	165	1320	$PGL_2(11)$
0	4	3	0	6	2	5616	44928	$2^{\bullet}[2^{\bullet}(PGL_3(3) : 2)]$
0	5	4	0	5	2	2040	16320	$2 \times [PGL_2(16) : 2]$
0	6	3	0	4	2	28224	225792	$[PGL_2(7) \times PGL_2(7)] : 2$

8.7 Progenitor $2^{*8} : [2^{2\bullet}2^2]$

$$2^{*8} : [2^{2\bullet}2^2] =< a, b, c, d, t | a^2, b^4, c^2, d^2, b^{-1}*a*b*a, b^{-1}*c*b^{-1}*d, (a*c)^2, (c*d)^2, a*b^{-2}*c*d, t^2, (t, d), (c*t^b)^j, (d*t)^k, (d*t^b)^l, (b*t)^m >$$

Table 8.7: Homomorphic Images of $2^{*8} : [2^{2\bullet}2^2]$

j	k	l	m	Index	Order of G	Isomorphic Type
0	0	0	3	12	192	$2^5 : S_3$
0	0	2	3	6	48	$2^2 : S_3$
0	0	3	5	120	1920	$2^4 \bullet [A_5 : 2]$
2	0	3	7	42	336	$PGL_2(7)$
2	0	3	9	612	4896	$PGL_2(17)$
2	0	4	5	40	320	$2^5 \bullet D_5$
2	0	5	5	90	720	$A_6 : 2$
3	8	3	5	15	120	$A_5 : 2$

8.8 Progenitor $2^{*11} : [D_{11}]$

$$2^{*11} : [11 : 2] =< a, b, t | b^2, (a^{(-1)}*b)^2, a^{(-11)}, t^2, (t, b),$$
$$(a*t)^i, (a^2*t*t^a)^j, (a^3*t*t^{(b*a)})^k, (t*a*t^{(a^3)}*a^4*t)^l >$$

Table 8.8: Homomorphic Images of $2^{*11} : [D_{11}]$

i	j	k	l	Index	Order of G	Isomorphic Type
0	2	0	4	7224	158928	$2 \times PGL_2(43)$
0	2	5	0	120	2640	$2 \times PGL_2(11)$
0	2	5	5	60	1320	$PGL_2(11)$
0	4	2	0	1104	24288	$2 \times PGL_2(23)$
7	2	0	0	3612	79464	$2 \times PSL_2(43)$

8.9 Progenitor $2^{*8} : [2^{2\bullet}2^2]$

$2^{*8} : [2^2 * 2^2] =< a, b, c, d, t | a^2, b^2, c^4, d^2, (a*b)^2, b*c^{(-1)}*b*c, c^{(-2)}*b*d, (a*d)^2, c^{(-1)}* b*a*c^{(-1)}*a, t^2, (t, a*b), (a*b*t^c)^k, (c*t)^l, (c^{(-1)}*t)^m, (a*c*t)^n, (c^{(-1)}*a*t)^o >$

Table 8.9: Homomorphic Images of $2^{*8} : 4^{\bullet}2^2$

i	k	l	m	n	Index	Order of G	Isomorphic Type
2	0	3	0	3	6	24	$2^2 : S_3$
3	0	0	0	5	120	1920	$2^4 \bullet [A_5 : 2]$
3	0	4	0	6	9	72	$3^{2\bullet}[2^{\bullet}2^2]$
3	0	5	0	6	15	120	$A_5 : 2$
3	0	6	0	10	480	7680	$2^6 \bullet [A_5 : 2]$
3	0	7	0	8	2688	43008	$2^7 \bullet PGL_2(7)$
5	0	4	0	10	25	200	$5^2 : [2^{\bullet}2^2]$
5	0	5	0	8	90	720	$A_6 : 2$

8.10 Progenitor $2^{*8} : [4 \times 2^2]$

$2^{*8} : [4 * 2^2] =< a, b, c, d, t | a^4, b^2, c^2, d^2, a^{(-2)}*b, (a^{(-1)}*c)^2, a^{(-1)}*d*a*d, d*c*a^2* d*c, t^2, (t, c), (c*a*t)^i, (d*a^{(-1)}*t)^k, (b*d*t^a)^l, (b*d*a*t)^m, (a*t)^n >$

8.11 Progenitor $2^{*4} : [D_4]$

$2^{*4} : [D_4] =< a, b, c, t | a^2, b^2, c^2, (a*b)^2, (b*c)^2, c*b*a*c*a, t^2, (t, c), (b*t)^e, (c*t^a* b*t^c)^g, (c*a*t*t^b)^h, (t^b*c*a*t*t^a)^i >$

Table 8.10: Homomorphic Images of $2^{*8} : 4^{\bullet}2^2$

i	k	l	m	n	Index	Order of G	Isomorphic Type
0	0	3	7	8	10752	172032	$2^9 \bullet PGL_2(7)$
0	0	3	7	9	8064	129024	$2^8 \bullet L_2(8)$
0	0	5	4	5	640	10240	$2^6 \bullet [2^4 : D_5]$
0	0	5	4	7	20160	322560	$2^{2\bullet}2^{\bullet}[M_{21} : 2]$
3	0	5	5	0	120	1920	$2^5 : A_5$
3	0	5	10	0	240	3840	$2^6 : A_5$
3	0	6	4	0	24	384	$2^5 : [2 \times S_3]$
3	0	6	6	3	27	108	$3^2 : [2 \times S_3]$
3	0	6	10	3	75	300	$5^2 : [2 \times S_3]$
3	0	7	8	3	84	336	$PGL_2(7)$
3	0	7	9	0	4032	64512	$2^7 \bullet L_2(8)$
3	0	7	9	3	126	504	$L_2(8)$
3	0	8	10	3	1080	4320	$[2 \times 3] : [A_6 : 2]$
3	0	9	9	3	855	3420	$L_2(19)$
3	0	9	10	3	5130	20520	$S_3 \times L_2(19)$
5	0	4	6	10	960	15360	$2^7 \bullet [A_5 : 2]$
5	0	4	9	5	1710	6840	$PGL_2(19)$
5	4	8	0	5	360	1440	$[2 \times A_6] : 2$
5	5	5	0	5	165	660	$L_2(11)$
6	0	7	8	3	10752	172032	$2^9 \bullet L_2(7)$
6	0	7	9	3	8064	129024	$2^8 \bullet L_2(8)$
8	0	3	8	10	129600	2073600	$2^{2\bullet}[(A_6 \times A_6) : 2^2]$

Table 8.11: Homomorphic Images of $2^{*4} : [2 : 2^2]$

e	g	h	i	Index	Order of G	Isomorphic Type
0	0	2	1	6	48	$2 \times S_4$
0	0	2	2	12	96	$2^4 : S_3$
0	0	2	3	18	144	$[3 \times 2^3] : S_3$
0	0	2	5	30	240	$[2^3 \times 5] : S_3$
0	0	2	6	36	288	$[2^4 \times 3] : S_3$
0	0	2	7	42	336	$[2^3 \times 7] : S_3$
0	0	3	2	84	672	$2^{\bullet}PGL_2(7)$
0	0	3	3	612	4896	$2^{\bullet}L_2(17)]$
0	3	0	3	81	648	$3^3 : S_4$
0	3	0	5	375	3000	$5^3 \bullet S_4$
0	3	0	7	1029	8232	$7^3 \bullet S_4$
0	3	10	10	750	6000	$5^3 : [2 \times S_4]$
0	4	0	2	120	960	$[2^{\bullet}2^2] \times [2 : A_5]$
0	4	0	3	12654	101232	$2 \times PGL_2(37)$
0	4	3	0	180	1440	$[2 \times A_6] : 2$
0	5	0	2	1710	13680	$2 \times PGL_2(19)$
0	6	3	0	3900	31200	$2 \times PGL_2(25)$
0	7	3	0	14112	112896	$2^{\bullet}[(L_2(7) \times L_2(7)) : 2]$
0	7	3	8	7056	56448	$[L_2(7) \times L_2(7)] : 2$
0	9	3	3	306	2448	$L_2(17)$
0	9	3	6	313344	2506752	$2^9 \bullet [2 \times L_2(17)]$
3	5	0	0	165	1320	$PGL_2(11)$
5	0	6	2	3600	28800	$2^{\bullet}[(A_5 \times A_5) : 2^2]$
5	4	0	4	7800	62400	$2^{\bullet}[PGL_2(25) : 2]$
5	4	6	0	2040	16320	$2^{\bullet}[L_2(16) : 2]$
5	4	8	0	885720	7085760	$2^{\bullet}[PGL_2(121) : 2]$
5	6	6	4	442860	3542880	$2 \times PGL_2(121)$
6	4	0	2	120	960	$[D_4 \times A_5] : 2$
6	8	4	3	90720	725760	$2^{\bullet}[A_9 : 2]$
6	8	7	2	20160	161280	$2^{\bullet}[M_{21} : 2^2]$
8	4	3	10	180	1440	$[2 \times A_5] : 2$
8	7	6	2	10752	86016	$2^{\bullet}[2^7 \bullet PGL_2(7)]$

Appendix A: MAGMA Code for Double Coset Enumeration of $[3 \times A_5] : 2$ over D_{10}

```
a:=0;b:=0;c:=3;d:=0;e:=3;f:=0;
G<x,y,t>:=Group<x,y,t|x^10,y^2,(x*y)^2,t^2,(t,y),
(x*t)^a, (x*y*t^x)^b, (x^2*y*t*t^x)^c, (t*t^x*t^(x^3))^d,
(x*t^x)^e,(y*t)^f>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
CompositionFactors(G1);

IN:=sub<G1|f(x),f(y)>;
S:=Sym(10);
xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10);
yy:=S!(2, 10)(3, 9)(4, 8)(5, 7);
N:=sub<S|xx,yy>;
#N;


xx*yy;
(1, 10)(2, 9)(3, 8)(4, 7)(5, 6)
xx^2;
(1, 3, 5, 7, 9)(2, 4, 6, 8, 10)
yy^2;
Id(S)
xx^3;
(1, 4, 7, 10, 3, 6, 9, 2, 5, 8)

Set(N);
```

```
ts:=[Id(G1): i in [1..10]];
ts[1]:=f(t); ts[2]:=f(t^x); ts[3]:=f(t^(x^2));
ts[4]:=f(t^(x^3)); ts[5]:=f(t^(x^4)); ts[6]:=f(t^(x^5));
ts[7]:=f(t^(x^6)); ts[8]:=f(t^(x^7)); ts[9]:=f(t^(x^8));
ts[10]:=f(t^(x^9));
f(x^3)*ts[4]*ts[3]*ts[2];

Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..20]];
for i in [2..20] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..20] do Sch[i], ArrayP[i]; end for;



Transitivity(N);

cst := [null : i in [1 .. 18]] where null is [Integers() | ];
prodim := function(pt, Q, I)
 /*
 Return the image of pt under permutations Q[I] applied
sequentially.
 */
 v := pt;
for i in I do
 v := v^(Q[i]);
 end for;
 return v;
 end function;

N1:=Stabiliser(N,1);
S:={[1]};
SS:=S^N;
```

```
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[1] eq n*(ts[(Rep(SSS[i]))[1]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

T1:=Transversal(N,N1);
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N1);

N12:=Stabiliser(N,[1,2]);
S:={[1,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[2] eq n*(ts[(Rep(SSS[i]))[1]]*
ts[(Rep(SSS[i]))[2]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

T12:=Transversal(N,N12);
for i in [1..#T12] do
ss:=[1,2]^T12[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

N12s:=sub<N|N12>;
for n in N do if 5^n eq 1 and 4^n eq 2
then N12s:=sub<N|N12s,n>; n; end if; end for;
```

```
#N12s, [1,2]^N12s;
T12:=Transversal(N,N12s);
for i in [1..#T12] do
ss:=[1,2]^T12[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N12s);

for g in IN do for h in IN do if ts[1]*ts[2] eq g*(ts[1])^
h then g,h; end if; end for; end for;

N13:=Stabiliser(N,[1,3]);
S:={[1,3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[3] eq n*(ts[(Rep(SSS[i]))[1]]*
ts[(Rep(SSS[i]))[2]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

N13s:=sub<N|N13>;
for n in N do if 6^n eq 1 and 8^n eq 3
then N13s:=sub<N|N13s,n>; n; end if; end for;
#N13s, [1,3]^N13s;
T13:=Transversal(N,N13s);
for i in [1..#T13] do
ss:=[1,3]^T13[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N13s);

N14:=Stabiliser(N,[1,4]);
S:={[1,4]};
SS:=S^N;
```

```
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[4] eq n*(ts[(Rep(SSS[i]))[1]]*
ts[(Rep(SSS[i]))[2]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

N14s:=sub<N|N14>;
for n in N do if 1^n eq 1 and 4^n eq 8
then N14s:=sub<N|N14s,n>; n; end if; end for;
#N14s, [1,4]^N14s;
T14:=Transversal(N,N14s);
for i in [1..#T14] do
ss:=[1,4]^T14[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N14s);

for g in IN do for h in IN do if ts[1]*ts[4]
eq g*(ts[1])^h then g,h; end if; end for; end for;

N15:=Stabiliser(N,[1,5]);
S:={[1,5]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[5] eq n*(ts[(Rep(SSS[i]))[1]]*
ts[(Rep(SSS[i]))[2]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

N15s:=sub<N|N15>;
for n in N do if 2^n eq 6 and 8^n eq 10
then N15s:=sub<N|N15s,n>; n; end if; end for;
#N15s, [1,5]^N15s;
```

```
T15:=Transversal(N,N15s);
for i in [1..#T15] do
ss:=[1,5]^T15[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N15s);

for g in IN do for h in IN do if ts[1]*ts[5]
eq g*(ts[1]*ts[3])^h then g,h; end if; end for; end for;

N16:=Stabiliser(N,[1,6]);
S:={[1,6]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[6] eq n*(ts[(Rep(SSS[i]))[1]]*
ts[(Rep(SSS[i]))[2]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

N16s:=sub<N|N16>;
for n in N do if 9^n eq 7 and 4^n eq 2
then N16s:=sub<N|N16s,n>; n; end if; end for;
#N16s, [1,6]^N16s;
T16:=Transversal(N,N16s);
for i in [1..#T16] do
ss:=[1,6]^T16[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N16s);

N161:=Stabiliser(N,[1,6,1]);
S:={[1,6,1]};
SS:=S^N;
SSS:=Setseq(SS);
```

```
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[6]*ts[1] eq n*(ts[(Rep(SSS[i]))[1]]*
ts[(Rep(SSS[i]))[2]]*ts[(Rep(SSS[i]))[3]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

N161s:=sub<N|N161>;
for n in N do if 1^n eq 6 and 6^n eq 1 and 1^n eq 6
then N161s:=sub<N|N161s,n>; n; end if; end for;
#N161s, [1,6,1]^N161s;
T161:=Transversal(N,N161s);
for i in [1..#T161] do
ss:=[1,6,1]^T161[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N161s);

for g in IN do for h in IN do if ts[1]*ts[6]*ts[1]
eq g*(ts[1]*ts[3])^h then g,h; end if; end for; end for;

N131:=Stabiliser(N,[1,3,1]);
S:={[1,3,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[3]*ts[1] eq n*(ts[(Rep(SSS[i]))[1]]*
ts[(Rep(SSS[i]))[2]]*ts[(Rep(SSS[i]))[3]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

N131s:=sub<N|N131>;

for n in N do if 3^n eq 9 and 1^n eq 7 and 3^n eq 9
then N131s:=sub<N|N131s,n>; n; end if; end for;
for n in N do if 3^n eq 9 and 1^n eq 1 and 3^n eq 9
```

```
then N131s:=sub<N|N131s,n>; n; end if; end for;
for n in N do if 3^n eq 5 and 1^n eq 3 and 3^n eq 5
then N131s:=sub<N|N131s,n>; n; end if; end for;

#N131s, [1,3,1]^N131s;
T131:=Transversal(N,N161s);
for i in [1..#T131] do
ss:=[1,3,1]^T131[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N131s);

for g in IN do for h in IN do if ts[1]*ts[3]*ts[1]
eq g*(ts[1]*ts[6])^h then g,h; end if; end for; end for;

N132:=Stabiliser(N,[1,3,2]);
S:={[1,3,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[3]*ts[2] eq n*(ts[(Rep(SSS[i]))[1]]*
ts[(Rep(SSS[i]))[2]]*ts[(Rep(SSS[i]))[3]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

N132s:=sub<N|N132>;
for n in N do if 1^n eq 5 and 3^n eq 3 and 2^n eq 4
then N132s:=sub<N|N132s,n>; n; end if; end for;
#N132s, [1,3,2]^N132s;
T132:=Transversal(N,N132s);
for i in [1..#T132] do
ss:=[1,3,2]^T132[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N132s);
```

```
for g in IN do for h in IN do if ts[1]*ts[3]*ts[2]
eq g*(ts[1])^h then g,h; end if; end for; end for;

N134:=Stabiliser(N,[1,3,4]);
S:={[1,3,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[3]*ts[4] eq n*(ts[(Rep(SSS[i]))[1]]*
ts[(Rep(SSS[i]))[2]]*ts[(Rep(SSS[i]))[3]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

N134s:=sub<N|N134>;
for n in N do if 1^n eq 1 and 3^n eq 9 and 4^n eq 8
then N134s:=sub<N|N134s,n>; n; end if; end for;
#N134s, [1,3,4]^N134s;
T134:=Transversal(N,N134s);
for i in [1..#T134] do
ss:=[1,3,4]^T134[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N134s);

for g in IN do for h in IN do if ts[1]*ts[3]*ts[4]
eq g*(ts[1])^h then g,h; end if; end for; end for;

N135:=Stabiliser(N,[1,3,5]);
S:={[1,3,5]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[3]*ts[5] eq n*(ts[(Rep(SSS[i]))[1]]*
ts[(Rep(SSS[i]))[2]]*ts[(Rep(SSS[i]))[3]])
then  print Rep(SSS[i]);
end if;
```

```
end for;
end for;

N135s:=sub<N|N135>;
for n in N do if 7^n eq 1 and 5^n eq 3 and 3^n eq 5
then N135s:=sub<N|N135s,n>; n; end if; end for;
#N135s, [1,3,5]^N135s;
T135:=Transversal(N,N135s);
for i in [1..#T135] do
ss:=[1,3,5]^T135[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..18] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N135s);

for g in IN do for h in IN do if ts[1]*ts[3]*ts[5] eq
g*(ts[1])^h then g,h; end if; end for; end for;
```

Appendix B: MAGMA Code for Extension Problem $U_3(3):2$

```
n:=0; o:=0; p:=0; q:=6; r:=0;
G<a,b,t>:=Group<a,b,t|a^2,b^3,(a*b)^7,(b*a*b^(-1)*a)^4,
t^2,(t,b),(t,a*b*a*b^-1*a),
(a*b^-1*a*t)^n,
(a*b^-1*a*t^a)^o,
(a*b^-1*a*t^(a^(b^2)))^p,
(a*b^-1*a*b*t)^q,
(a*b^-1*a*b*t^(a^(b^2)))^r>;
#G;

#DoubleCosets(G,sub<G|a,b>,sub<G|a,b>);

f,G1,k:=CosetAction(G,sub<G|a,b>);

#k;

CompositionFactors(G1);

NL:=NormalLattice(G1);
NL;

for g,h in NL[2] do if Order(g) eq 2 and Order(h) eq 6 and NL[2]
eq sub<NL[2]|g,h>then a:=g; b:=h; break;
end if;
end for;

A:=G1!(1, 21)(3, 25)(4, 27)(6, 44)(9, 55)(11, 60)(13, 22)
    (14, 53)(15, 63)(17, 64)(18, 30)(19, 43)(20, 70)(23, 68)
    (28, 36)(31, 59)(33, 35)(37, 72)(41, 58)(42, 65)(46, 56)
    (47, 62)(48, 66)(50, 52);
```

```
B:=G1!(1, 23, 62, 5, 59, 40)(2, 49, 66)(3, 50, 39)(4, 33,
    43, 25, 53, 14)(6, 61, 56, 26, 72, 48)(7, 51, 64, 21,
    29, 41)(8, 68, 71)(9, 37, 16, 19, 54, 12)(10, 70, 22)
    (11, 57, 47, 17, 32, 38)(13, 31, 18)(15, 63, 65, 60,
    58, 20)(24, 35, 45, 44, 55, 34)(27, 67, 28)(30, 42, 69)
    (36, 52, 46);

NL[2] eq sub<NL[2]|a,b>;

for r in NL[3] do if Order(r) eq 2 and r notin NL[2] and NL[3]
eq sub<NL[3]|NL[2],r>
then R:=r; break; end if; end for;

R:=G1!(1, 6)(2, 15)(3, 32)(4, 8)(5, 61)(7, 28)(9, 42)(10, 44)
    (11, 50)(12, 17)(13,46)(14, 20)(16, 69)(18, 36)(19, 38)
    (21, 45)(22, 35)(23, 26)(24, 41)(25, 63)(27, 64)(29, 67)
    (30, 54)(31, 52)(33, 60)(34, 70)(37, 57)(39, 47)(40,48)
    (43, 68)(49, 65)(51, 55)(53, 71)(56, 59)(58, 66)(62, 72);

NN<i,j>:=Group<i,j|i^2,j^6,(i*j)^7,(i,(i*j^2)^3),
j^3*(j^2,i*j^3*i)^2>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..6048]];
for i in [2..6048] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

AA:=[Id(NN) : i in [1..2]];
for i in [1..6048] do if A^R eq ArrayP[i] then AA[1]:=Sch[i];
Sch[i]; end if; end for;

for i in [1..6048] do if B^R eq ArrayP[i] then AA[1]:=Sch[i];
Sch[i]; end if; end for;
```

```
H<i,j,k>:=Group<i,j,k|i^2,j^6,(i*j)^7,(i,(i*j^2)^3),
j^3*(j^2,i*j^3*i)^2,
k^2,i^k=i * j * i * j * i * j^3 * i * j^-1 * i * j^-1 * i,
j^k=i * j * i * j^-2 * i * j^-1 * i * j^-2 * i * j^-1 * i *
j^-1 * i * j>;
f1,h,k1:=CosetAction(H,sub<H|Id(H)>);
s:=IsIsomorphic(NL[3],h);
s;
```

Appendix C: MAGMA Code for Extension Problem Method 1

$2 \times [A_6 : 3]$

```
i:=0;j:=3;k:=0;l:=0;m:=10;
G<a,b,x,t>:=Group<a,b,x,t|a^4,b^4,(a,b),x^2,a^x=b,b^x=a,
t^2,(t,b),(x*t)^i,(x*a*t*t^x)^j,(a*t*t^a)^k,
(a*x*t^(x^2)*t^x*t)^l,(a*x*t^(x^2)*t^a*t)^m>;
#G;

Index(G,sub<G|a,b,x>);

f,G1,k:=CosetAction(G,sub<G|a,b,x>);
#k;

CompositionFactors(G1);

NL:=NormalLattice(G1);
NL;

Center(G1);

for i in [6..8] do if IsIsomorphic(DirectProduct(NL[2],
NL[i]),G1)
then i; end if; end for;

CompositionFactors(NL[8]);
nl:=NormalLattice(NL[8]);
nl;
Center(NL[8]);
```

```
S:=Subgroups(NL[8]);
S;
D:=DerivedGroup(NL[8]);
D;
CompositionFactors(D);
DB:= PerfectGroupDatabase();
DB;
for i in [1..2] do Group(DB,1080,i); end for;
s:=IsIsomorphic(PermutationGroup(DB,1080,1),nl[3]);
s;

H<a,b,c>:=Group<a,b,c|a^6,b^3,c^3,(b*c)^4,(b*c^-1)^5,
a^-1*b^-1 * c * b * c * b^-1 * c * b * c^-1>;
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
#H1;
s,t :=IsIsomorphic(H1,nl[3]);
s;

a:=t(f1(a)); b:=t(f1(b)); c:=t(f1(c));

for g in NL[8] do if Order(g) eq 2 and NL[8] eq
sub<NL[8]|nl[3],g> then U:=g; break; end if; end for;

U;

Order(U);
NL[8] eq sub<NL[8]|nl[3],U>;

N:=sub<NL[8]|a,b,c>;
#N;
NN<i,j,k>:=Group<i,j,k|i^6,j^3,k^3,(j*k)^4,(j*k^-1)^5,
i^-1*j^-1 * k * j * k *j^-1 * k * j * k^-1>;
#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..1080]];
for i in [2..1080] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do

if Eltseq(Sch[i])[j] eq 1
then P[j]:=a; end if;

if Eltseq(Sch[i])[j] eq -1
```

```
then P[j]:=a^-1; end if;


if Eltseq(Sch[i])[j] eq 2
then P[j]:=b; end if;

if Eltseq(Sch[i])[j] eq -2
then P[j]:=b^-1; end if;

if Eltseq(Sch[i])[j] eq 3
then P[j]:=c; end if;

if Eltseq(Sch[i])[j] eq -3
then P[j]:=c^-1; end if;

end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

A:=[Id(NN) : i in [1..2]];
for i in [1..1080] do if a^U eq ArrayP[i] then A[1]:=Sch[i];
Sch[i]; end if;end for;
for i in [1..1080] do if b^U eq ArrayP[i] then A[1]:=Sch[i];
Sch[i]; end if;end for;
for i in [1..1080] do if c^U eq ArrayP[i] then A[1]:=Sch[i];
Sch[i]; end if;end for;

HH<a,b,c,u>:=Group<a,b,c,u|a^6,b^3,c^3,(b*c)^4,(b*c^-1)^5,
a^-1*b^-1 * c * b * c * b^-1 * c * b * c^-1, u^2,
a^u=b*a^-1 *b^-1, b^u=a*c^-1*a, c^u=c^-1*b^-1*c>;
f2,H2,k2:=CosetAction(HH,sub<HH|Id(HH)>);
#H2;
s,t :=IsIsomorphic(H2,NL[8]);
s;

/*Lastly, add the action of Cyclic(2) into the presentation of Alt(6) and
Cyclic(3).*/

HHG<a,b,c,u,d>:=Group<a,b,c,u,d|a^6,b^3,c^3,(b*c)^4,(b*c^-1)^5,
a^-1*b^-1 * c * b * c * b^-1 * c * b * c^-1, u^2,
a^u=b*a^-1 *b^-1, b^u=a*c^-1*a, c^u=c^-1*b^-1*c,d^2,
```

```
(a,d),(b,d),(c,d),(u,d)>;
f3,H3,k3:=CosetAction(HHG,sub<HHG|Id(HHG)>);
#H3;
s,t :=IsIsomorphic(H3,G1);
s;
```

Appendix D: MAGMA Code for Extension Problem Method 2

$6 \bullet PGL_2(9)$

```
i:=0;j:=3;k:=0;l:=0;m:=10;
G<a,b,x,t>:=Group<a,b,x,t|a^4,b^4,(a,b),x^2,a^x=b,b^x=a,
t^2,(t,b),(x*t)^i,(x*a*t*t^x)^j,(a*t*t^a)^k,
(a*x*t^(x^2)*t^x*t)^l,(a*x*t^(x^2)*t^a*t)^m>;
#G;
f,G1,k:=CosetAction(G,sub<G|a,b,x>);
NL:=NormalLattice(G1);
NL;
NL[4];
Order(NL[4].1);
D:=NL[4].1;
NL[4] eq sub<NL[4]|D>;
/* NL[4] is a cyclic group of order 6 generated by D */
IsCyclic(NL[4]);
q:=quo<G1|NL[4]>;
P:=PGL(2,9);
s:=IsIsomorphic(q,P);
s;
  FPGroup(q);
  H<g,h,i,j>:=Group<g,h,i,j|g^2,h^2,i^2,j^2, (g*h)^2,(h*j)^2,
  g * i * h * i,
j * g * j * h * g * j * g * j * g,
  (g * j * i * j)^3,
j * i * g * j *i * j * i * g * j * i * h * j * i* j*
g * i * j * i * j * i>;
  #H;
  f1,h1,k1:=CosetAction(H,sub<H|Id(H)>);
```

```
s:=IsIsomorphic(q,h1);
s;
T:=Transversal(G1,NL[4]);
T2:=T[2]; T3:=T[3];T4:=T[4];T5:=T[5];

g:=T[2]; h:=T[3]; i:=T[4]; j:=T[5];
Order(j * g * j * h * g * j * g * j * g);

Order(j * i * g * j *i * j * i * g * j * i * h *
 j * i* j* g * i * j * i * j* i);

for m in [1..6] do if D^g eq D^m then m; end if; end for;
for m in [1..6] do if D^h eq D^m then m; end if; end for;
for m in [1..6] do if D^i eq D^m then m; end if; end for;
for m in [1..6] do if D^j eq D^m then m; end if; end for;
/* We note that D^g=D, D^h=D, D^i=D, and D^j=D^5. */
G<d,g,h,i,j>:=Group<d,g,h,i,j|d^6,g^2,h^2,i^2,j^2,
(g*h)^2,(h*j)^2,
g * i * h * i,
j * g * j * h * g * j * g * j * g=d^3,  (g * j * i * j)^3,
j * i * g * j *i * j * i * g * j * i * h *
j * i* j* g * i * j * i * j *
i=d^4,
d^g=d, d^h=d, d^i=d,d^j=d^5>;
#G;
f2,g2,k2:=CosetAction(G,sub<G|Id(G)>);
#g2;
s:=IsIsomorphic(G1,g2);
s;
```

Appendix E: MAGMA Code for Wreath Product Progenitor $2^{*8} : \mathbb{Z}_4 \wr \mathbb{Z}_2$

```
W:=WreathProduct(CyclicGroup(4),CyclicGroup(2));

S:=Sym(8);
A:=S!(1,3,5,7);
B:=S!(2,4,6,8);
X:=S!(1,2)(3,4)(5,6)(7,8);
N:=sub<S|A,B,X>;
s:=IsIsomorphic(N,W);
s;
G<a,b,x>:=Group<a,b,x|a^4,b^4,(a,b),x^2,a^x=b,b^x=a>;

f,G1,k:=CosetAction(G,sub<G|Id(G)>);
z:=IsIsomorphic(G1,W);
z;

NN<a,b,x>:=Group<a,b,x|a^4,b^4,(a,b),x^2,a^x=b,b^x=a>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..32]];
for i in [2..32] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=X; end if;
```

```
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

N1:=Stabiliser(N,1);
N1;

Centraliser(N,Stabiliser(N,[1,2]));

for i,j,k,l,m in [0..10] do
G<a,b,x,t>:=Group<a,b,x,t|a^4,b^4,(a,b),x^2,a^x=b,b^x=a,
t^2,(t,b),(x*t)^i,(x*a*t*t^x)^j,(a*t*t^a)^k,
(a*x*t^(x^2)*t^x*t)^l,(a*x*t^(x^2)*t^a*t)^m>;
i,j,k,l,m, #G;
end for;

for i,j,k,l,m in [0..10] do
G<a,b,x,t>:=Group<a,b,x,t|a^4,b^4,(a,b),x^2,a^x=b,b^x=a,
t^2,(t,b),(x*t)^i,(x*a*t*t^x)^j,(a*t*t^a)^k,
(a*x*t^(x^2)*t^x*t)^l,(a*x*t^(x^2)*t^a*t)^m>;
if Index(G,sub<G|a,b,x>) ge 3 then i,j,k,l,m,
Index(G,sub<G|a,b,x>);
end if; end for;

nano 2star8Z4WrZ2
nohup magma "2star8Z4WrZ2" &> 2star8Z4WrZ2.out &
nano 2star8Z4WrZ2.out
```

Appendix F: MAGMA Code for Wreath Product Progenitor $2^{*10} : \mathbb{Z}_2 \wr S_5$

```
W:=WreathProduct(CyclicGroup(2),Sym(5));

S:=Sym(10);
A:=S!(1,6);
B:=S!(2,7);
C:=S!(3,8);
D:=S!(4,9);
E:=S!(5,10);
X:=S!(1,2,3,4,5)(6,7,8,9,10);
Y:=S!(1,2)(6,7);
N:=sub<S|A,B,C,D,E,X,Y>;
s:=IsIsomorphic(N,W);
s;
G<a,b,c,d,e,x,y>:=Group<a,b,c,d,e,x,y|a^2,b^2,c^2,d^2,e^2,
(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),
x^5,y^2,(x*y)^4,(x,y)^3,a^x=b,b^x=c,c^x=d,
d^x=e,e^x=a,a^y=b,b^y=a,c^y=c,d^y=d,e^y=e>;

f,G1,k:=CosetAction(G,sub<G|Id(G)>);
z:=IsIsomorphic(G1,W);
z;

NN<a,b,c,d,e,x,y>:=Group<a,b,c,d,e,x,y|a^2,b^2,c^2,d^2,e^2,
(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),
x^5,y^2,(x*y)^4,(x,y)^3,a^x=b,b^x=c,c^x=d,
d^x=e,e^x=a,a^y=b,b^y=a,c^y=c,d^y=d,e^y=e>;
#NN;
```

```
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..3840]];
for i in [2..3840] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=X; end if;
if Eltseq(Sch[i])[j] eq -6 then P[j]:=X^-1; end if;
if Eltseq(Sch[i])[j] eq 7 then P[j]:=Y; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

N1:=Stabiliser(N,1);
N1;

for i in [1..3840] do if ArrayP[i] eq N!(4, 10)(5, 9)
then print Sch[i];end if; end for;

for i in [1..3840] do if ArrayP[i] eq N!(3, 4, 10)(5, 8, 9)
then print Sch[i];end if; end for;

for i in [1..3840] do if ArrayP[i] eq N!(2, 10, 9)(4, 7, 5)
then print Sch[i];end if; end for;

for i in [1..3840] do if ArrayP[i] eq N!(2, 10)(5, 7)
then print Sch[i];end if; end for;

for i,j,k,l,m in [0..10] do
G<a,b,c,d,e,x,y,t>:=Group<a,b,c,d,e,x,y,t|a^2,b^2,c^2,d^2,e^2,
(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),
x^5,y^2,(x*y)^4,(x,y)^3,a^x=b,b^x=c,c^x=d,
d^x=e,e^x=a,a^y=b,b^y=a,c^y=c,d^y=d,e^y=e,
t^2,(t,b),(t,c),(t,d),(t,e),(t,d*e*x^2*y*x^-2),
(t,d*e*x*y*x^-1*y*x),(t,b*d*y*x^-2*y*x),(t,b*e*y*x*y*x^-1*y),
(a*t^a)^i,
(x*a*t*t^x)^j,
```

```
(y*a*t*t^y)^k,
(a*x*y*t^(x^2)*t^x*t^y)^l,
(a*x*y*t^(x^2)*t^a*t)^m>;
i,j,k,l,m, #G;
end for;

for i,j,k,l,m in [0..10] do
G<a,b,c,d,e,x,y,t>:=Group<a,b,c,d,e,x,y,t|a^2,b^2,c^2,d^2,e^2,
(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),
x^5,y^2,(x*y)^4,(x,y)^3,a^x=b,b^x=c,c^x=d,
d^x=e,e^x=a,a^y=b,b^y=a,c^y=c,d^y=d,e^y=e,
t^2,(t,b),(t,c),(t,d),(t,e),(t,d*e*x^2*y*x^-2),
(t,d*e*x*y*x^-1*y*x),(t,b*d*y*x^-2*y*x),(t,b*e*y*x*y*x^-1*y),
(a*t^a)^i,
(x*a*t*t^x)^j,
(y*a*t*t^y)^k,
(a*x*y*t^(x^2)*t^x*t^y)^l,
(a*x*y*t^(x^2)*t^a*t)^m>;
if Index(G,sub<G|a,b,c,d,e,x,y>) ge 3 then i,j,k,l,m,
Index(G,sub<G|a,b,c,d,e,x,y>);
end if; end for;

nano 2star10Z2WrS5
nohup magma "2star10Z2WrS5" &> 2star10Z2WrS5.out &
nano 2star10Z2WrS5.out
```

Appendix G: MAGMA Code for Wreath Product Progenitor $2^{*12} : \mathbb{Z}_3 \wr S_4$

```
W:=WreathProduct(CyclicGroup(3),Sym(4));

S:=Sym(12);
A:=S!(1,5,9);
B:=S!(2,6,10);
C:=S!(3,7,11);
D:=S!(4,8,12);
X:=S!(1,2,3,4)(5,6,7,8)(9,10,11,12);
Y:=S!(1,2)(5,6)(9,10);
N:=sub<S|A,B,C,D,X,Y>;
s:=IsIsomorphic(N,W);
s;
G<a,b,c,d,x,y>:=Group<a,b,c,d,x,y|a^3,b^3,c^3,d^3,(a,b),(a,c),
(a,d),(b,c),(b,d),(c,d),x^4,y^2,(x*y)^3,a^x=b,b^x=c,c^x=d,
d^x=a,a^y=b,b^y=a,c^y=c,d^y=d>;

f,G1,k:=CosetAction(G,sub<G|Id(G)>);
z:=IsIsomorphic(G1,W);
z;

NN<a,b,c,d,x,y>:=Group<a,b,c,d,x,y|a^3,b^3,c^3,d^3,(a,b),(a,c),
(a,d),(b,c),(b,d),(c,d),x^4,y^2,(x*y)^3,a^x=b,b^x=c,c^x=d,
d^x=a,a^y=b,b^y=a,c^y=c,d^y=d>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..1944]];
for i in [2..1944] do
```

```
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=C^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=D^-1; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=X; end if;
if Eltseq(Sch[i])[j] eq -5 then P[j]:=X^-1; end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=Y; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

N1:=Stabiliser(N,1);
N1;

for i in [1..1944] do if ArrayP[i] eq N!(2, 3)(4, 12, 8)(6, 7)
(10, 11) then print Sch[i];end if; end for;

for i in [1..1944] do if ArrayP[i] eq N!(2, 3, 12, 10, 11,
8, 6, 7, 4) then print Sch[i];end if; end for;

for i in [1..1944] do if ArrayP[i] eq N!(2, 4, 10, 12, 6, 8)
then print Sch[i];end if; end for;

for i,j,k,l,m in [0..10] do
G<a,b,c,d,x,y,t>:=Group<a,b,c,d,x,y,t|a^3,b^3,c^3,d^3,(a,b),
(a,c),(a,d),(b,c),(b,d),(c,d),x^4,y^2,(x*y)^3,a^x=b,b^x=c,c^x=d,
d^x=a,a^y=b,b^y=a,c^y=c,d^y=d,t^2,(t,b),(t,c),(t,d),
(t,d^-1*x^-1*y*x),(t,x*y*d^-1),(t,y*x^2*y*x*b^-1),
(a*t^a)^i,
(x*a*t*t^x)^j,
(y*a*t*t^y)^k,
(a*x*y*t^(a^2)*t^x*t^y)^l,
(a*x*y*t^(x^2)*t^a*t)^m>;
if Index(G,sub<G|a,b,c,d,x,y>) ge 3 then i,j,k,l,m,
Index(G,sub<G|a,b,c,d,x,y>);
```

```
end if; end for;

nano 2star12Z3WrS4
nohup magma "2star12Z3WrS4" &> 2star12Z3WrS4.out &
nano 2star12Z3WrS4.out
```

Appendix H: MAGMA Code for Wreath Product Progenitor $2^{*14} : \mathbb{Z}_7 \wr \mathbb{Z}_2$

```
W:=WreathProduct(CyclicGroup(7),CyclicGroup(2));

S:=Sym(14);
A:=S!(1,3,5,7,9,11,13);
B:=S!(2,4,6,8,10,12,14);
X:=S!(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14);
N:=sub<S|A,B,X>;
s:=IsIsomorphic(N,W);
s;
G<a,b,x>:=Group<a,b,x|a^7,b^7,(a,b),x^2,a^x=b,b^x=a>;

f,G1,k:=CosetAction(G,sub<G|Id(G)>);
z:=IsIsomorphic(G1,W);
z;

NN<a,b,x>:=Group<a,b,x|a^7,b^7,(a,b),x^2,a^x=b,b^x=a>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..98]];
for i in [2..98] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=X; end if;
```

```
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

N1:=Stabiliser(N,1);
N1;

for i,j,k,l,m in [0..10] do
G<a,b,x,t>:=Group<a,b,x,t|a^7,b^7,(a,b),x^2,a^x=b,b^x=a,
t^2,(t,b),(x*t*a)^i,(x*t^a)^j,(a*t*t^a^2)^k,(x*a*t*a^2)^l,
(a^3*t^x*t^a)^m>;
i,j,k,l,m, #G;
end for;

for i,j,k,l,m in [0..10] do
G<a,b,x,t>:=Group<a,b,x,t|a^7,b^7,(a,b),x^2,a^x=b,b^x=a,
t^2,(t,b),(x*t*a)^i,(x*t^a)^j,(a*t*t^a^2)^k,(x*a*t*a^2)^l,
(a^3*t^x*t^a)^m>;
if Index(G,sub<G|a,b,x>) ge 3 then i,j,k,l,m,
Index(G,sub<G|a,b,x>);
end if; end for;

nano 2star14Z7WrZ2
nohup magma "2star14Z7WrZ2" &> 2star14Z7WrZ2.out &
nano 2star14Z7WrZ2.out
```

Appendix I: MAGMA Code for Wreath Product Progenitor $2^{*15} : \mathbb{Z}_5 \wr \mathbb{Z}_3$

```
W:=WreathProduct(CyclicGroup(5),CyclicGroup(3));

S:=Sym(15);
A:=S!(1,4,7,10,13);
B:=S!(2,5,8,11,14);
C:=S!(3,6,9,12,15);
X:=S!(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15);
N:=sub<S|A,B,C,X>;
s:=IsIsomorphic(N,W);
s;
G<a,b,c,x>:=Group<a,b,c,x|a^5,b^5,c^5,(a,b),(a,c),(b,c),x^3,
a^x=b,b^x=c,c^x=a>;

f,G1,k:=CosetAction(G,sub<G|Id(G)>);
z:=IsIsomorphic(G1,W);
z;

NN<a,b,c,x>:=Group<a,b,c,x|a^5,b^5,c^5,(a,b),(a,c),(b,c),x^3,
a^x=b,b^x=c,c^x=a>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..375]];
for i in [2..375] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
```

```
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=C^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=X; end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=X^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

N1:=Stabiliser(N,1);
N1;

for i,j,k,l,m in [0..10] do
G<a,b,c,x,t>:=Group<a,b,c,x,t|a^5,b^5,c^5,(a,b),(a,c),(b,c),x^3,
a^x=b,b^x=c,c^x=a,t^2,(t,b),(t,c),
(a*t^a)^i,(x*a*t*t^x)^j,(a*t*t^a)^k,(a*x*t^(x^2)*t^x*t)^l,
(a*x*t^(x^2)*t^a*t)^m>;
i,j,k,l,m, #G;
end for;

for i,j,k,l,m in [0..10] do
G<a,b,c,x,t>:=Group<a,b,c,x,t|a^5,b^5,c^5,(a,b),(a,c),(b,c),x^3,
a^x=b,b^x=c,c^x=a,t^2,(t,b),(t,c),
(a*t^a)^i,(x*a*t*t^x)^j,(a*t*t^a)^k,(a*x*t^(x^2)*t^x*t)^l,
(a*x*t^(x^2)*t^a*t)^m>;
if Index(G,sub<G|a,b,c,x>) ge 3 then i,j,k,l,m,
Index(G,sub<G|a,b,c,x>);
end if; end for;

nano 2star15Z5WrZ3
nohup magma "2star15Z5WrZ3" &> 2star15Z5WrZ3.out &
nano 2star15Z5WrZ3.out
```

Appendix J: MAGMA Code for Monomial Progenitor $3^{*3} :_m L_2(3)$

```
S:=Sym(4);
xx:=S!(1,3,4);
yy:=S!(1,2)(3,4);
G:=sub<S|xx,yy>;
C:=Classes(G); C;

S:=Subgroups(G);
#S;
Cr:=CharacterTable(G);
for i in [1..#Cr] do i,Cr[i][1];
end for;
for i in [1..#S] do i, Index(G,S[i]`subgroup);
end for;

N:=S[4]`subgroup;

cn:=Classes(N); cn;
CN:=CharacterTable(N); CN;
CT:=CharacterTable(G);
ind:=Induction(CN[2],G);
ind;
Norm(ind);
T:=Transversal(G,N);
T; #T;

for i in [1..3] do if xx*T[i]^-1 in N then i,
CN[2](xx*T[i]^-1);end if; end for;

for i in [1..3] do if T[2]*xx*T[i]^-1 in N then i,
CN[2](T[2]*xx*T[i]^-1);end if; end for;
```

```
for i in [1..3] do if T[3]*xx*T[i]^-1 in N then i,
CN[2](T[3]*xx*T[i]^-1);end if; end for;

for i in [1..3] do if yy*T[i]^-1 in N then i,
CN[2](yy*T[i]^-1);end if; end for;

for i in [1..3] do if T[2]*yy*T[i]^-1 in N then i,
CN[2](T[2]*yy*T[i]^-1);end if; end for;

for i in [1..3] do if T[3]*yy*T[i]^-1 in N then i,
CN[2](T[3]*yy*T[i]^-1);end if; end for;

S:=Sym(6);
A:=S!(1,2,3)(4,5,6);
B:=S!(2,5)(3,6);
N:=sub<S|A,B>;

G<x,y>:=Group<x,y|x^3,y^2,(x*y)^3>;

NN<x,y>:=Group<x,y|x^3,y^2,(x*y)^3>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..12]];
for i in [2..12] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

Ngt1:=Stabiliser(N,{1,4});
H:=sub<N|Id(N)>;
for g in N do if {1,4}^g eq {1,4} then H:=sub<N|H,g>;
end if; end for;
H eq Ngt1; Ngt1;

for i in [1..12] do Sch[i], ArrayP[i]; end for;
```

```
C:=Classes(N);
C;

C2:=Centraliser(N,N!(1, 4)(2, 5));
C2;
Orbits(C2);
C3:=Centraliser(N,N!(1, 2, 3)(4, 5, 6));
C3;
Orbits(C3);
C4:=Centraliser(N,N!(1, 3, 2)(4, 6, 5));
C4;
Orbits(C4);

(x*y*x^(-1)*t)^a
(x*y*x^(-1)*t^x)^b
(x*y*x^(-1)*t^(x^(-1)))^c

(x*t)^d
(x*t^(y^x))^e

(x^(-1)*t)^f **
(x^(-1)*t^(y^x))^g

for a,d,f,h,i in [0..5] do
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^3,t^2,(t,y),
t^(x*y*x^(-1))=t^(-1),
(y*(t^y)^x*t*t^x)^a,
(t^y*x*y*t)^d,
(x*t*t^x)^f,
(t*t^x*t^y*x)^h,
(x*t)^i>;
a,d,f,h,i, #G;
end for;

for a,d,f,h,i in [0..15] do
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^3,t^2,(t,y),
t^(x*y*x^(-1))=t^(-1),
(y*(t^y)^x*t*t^x)^a,
(t^y*x*y*t)^d,
(x*t*t^x)^f,
(t*t^x*t^y*x)^h,
(x*t)^i>;
if Index(G,sub<G|x,y>) ge 3 then a,d,f,h,i,
```

```
Index(G,sub<G|x,y>),#G;
end if; end for;

nano 3star4mL23b
nohup magma "3star4mL23b" &> 3star4mL23b.out &
nano 3star4mL23b.out
```

Appendix K: MAGMA Code for Transitive Progenitor $2^{*8} : [2^{\bullet}(2^{2\bullet}2^{2})]$

```
NumberOfTransitiveGroups(8);
/* for i in [1..50] do #TransitiveGroup(8, i); end for; */
N:=TransitiveGroup(8,19);
#N;
IdentifyGroup(N);
D:=SmallGroupDatabase ();
G:=SmallGroup(D,32,6);
G;
FPGroup(G);
G<a,b,c,d,e>:=Group<a,b,c,d,e|a^2=d,b^2,c^2,d^2,e^2,b^a=b*c,
c^a=c*e,c^b=c,d^a=d,d^b=d*e,d^c=d,e^a=e,e^b=e,e^c=e,e^d=e>;
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
SL := Subgroups(G1);
T := {X`subgroup: X in SL};
#T;
TrivCore := {H:H in T| #Core(G1,H) eq 1};
#TrivCore;
TrivCore;
mdeg := Min({Index(G1,H):H in TrivCore});
Good := {H: H in TrivCore| Index(G1,H) eq mdeg};
#Good;
H := Rep(Good);
#H;
f,G1,K := CosetAction(G1,H);
G1;
s:=IsIsomorphic(G1,N);
s;
```

```
G1;
S:=Sym(8);
A:=S!(2, 5, 6, 8)(3, 7);
B:=S!(1, 2)(3, 5)(4, 6)(7, 8);
C:=S!(1, 3)(2, 5)(4, 7)(6, 8);
D:=S!(2, 6)(5, 8);
E:=S!(1, 4)(2, 6)(3, 7)(5, 8);
N:=sub<S|A,B,C,D>;
N eq G1;
N1:=Stabiliser(N,1);
N1;

for i,j,k,l,m in [0..10] do
G<a,b,c,d,e,t>:=Group<a,b,c,d,e,t|a^2=d,b^2,c^2,d^2,e^2,b^a=b*c,
c^a=c*e,c^b=c,d^a=d,d^b=d*e,d^c=d,e^a=e,e^b=e,e^c=e,e^d=e,
t^2,(t,a),(t,d),
(c*t^b)^i,(b*e*t*t^b)^j,(c*e*t*t^b)^k,(b*c*e*t^(b^2)*t^c*t^e)^l,
(b*c*e*t^(b^2)*t^c*t)^m>;
i,j,k,l,m, #G;
end for;

for i,j,k,l,m in [0..10] do
G<a,b,c,d,e,t>:=Group<a,b,c,d,e,t|a^2=d,b^2,c^2,d^2,e^2,b^a=b*c,
c^a=c*e,c^b=c,d^a=d,d^b=d*e,d^c=d,e^a=e,e^b=e,e^c=e,e^d=e,
t^2,(t,a),(t,d),
(c*t^b)^i,(b*e*t*t^b)^j,(c*e*t*t^b)^k,(b*c*e*t^(b^2)*t^c*t^e)^l,
(b*c*e*t^(b^2)*t^c*t)^m>;
if Index(G,sub<G|a,b,c,d,e>) ge 3 then i,j,k,l,m,
Index(G,sub<G|a,b,c,d,e>);
end if; end for;

nano 2star8N32
nohup magma "2star8N32" &> 2star8N32.out &
nano 2star8N32.out
```

Appendix L: MAGMA Code for Transitive Progenitor $2^{*10} : [2^4 : 5]$

```
NumberOfTransitiveGroups(10);
N:=TransitiveGroup(10,8);
#N;
IdentifyGroup(N);
D:=SmallGroupDatabase ();
G:=SmallGroup(D,80,49);
G;
FPGroup(G);
G<a,b,c,d,e>:=Group<a,b,c,d,e|a^5,b^2,c^2,d^2,e^2,b^a=e,c^a=b*e,
c^b=c,d^a=b*c*e,d^b=d,d^c=d,e^a=b*c*d*e,e^b=e,e^c=e,e^d=e>;
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
SL := Subgroups(G1);
T := {X`subgroup: X in SL};
#T;
TrivCore := {H:H in T| #Core(G1,H) eq 1};
#TrivCore;
TrivCore;
mdeg := Min({Index(G1,H):H in TrivCore});
Good := {H: H in TrivCore| Index(G1,H) eq mdeg};
#Good;
H := Rep(Good);
#H;
f,G1,K := CosetAction(G1,H);
G1;
s:=IsIsomorphic(G1,N);
s;
G1;
S:=Sym(10);
A:=S!(1, 2, 4, 6, 8)(3, 5, 7, 9, 10);
B:=S!(1, 3)(2, 5)(4, 7)(8, 10);
```

```
C:=S!(4, 7)(6, 9);
D:=S!(2, 5)(6, 9);
E:=S!(1, 3)(2, 5)(4, 7)(6, 9);
N:=sub<S|A,B,C,D,E>;
N eq G1;
N1:=Stabiliser(N,1);
N1;

NN<a,b,c,d,e>:=Group<a,b,c,d,e|a^5,b^2,c^2,d^2,e^2,b^a=e,c^a=b*e,
c^b=c,d^a=b*c*e,d^b=d,d^c=d,e^a=b*c*d*e,e^b=e,e^c=e,e^d=e>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..80]];
for i in [2..80] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

N1:=Stabiliser(N,1);
N1;

for i in [1..80] do if ArrayP[i] eq N!(4, 7)(8, 10)
then print Sch[i];end if; end for;

for i,j,k,l,m in [0..10] do
G<a,b,c,d,e,t>:=Group<a,b,c,d,e,t|a^5,b^2,c^2,d^2,e^2,b^a=e,
c^a=b*e,c^b=c,d^a=b*c*e,d^b=d,d^c=d,e^a=b*c*d*e,e^b=e,e^c=e,
e^d=e,t^2,(t,c),(t,d),(t,b*c*e),
(a*t^b)^i,(a^2*e*t*t^b)^j,(a*e*t*t^b)^k,(b*a*e*t^(b^2)*t^a*t^e)^l,
(b*a^2*e*t^(a^2)*t^a*t)^m>;
if Index(G,sub<G|a,b,c,d,e>) ge 3 then i,j,k,l,m,
Index(G,sub<G|a,b,c,d,e>);
end if; end for;
```

```
i,j,k,l,m, #G;
end for;

nano 2star10N80
nohup magma "2star10N80" &> 2star10N80.out &
nano 2star10N80.out
```

Appendix M: MAGMA Code for Transitive Progenitor $2^{*10} : [2^4 : 5]$

```
NumberOfTransitiveGroups(7);
N:=TransitiveGroup(7,5);
#N;
IdentifyGroup(N);
D:=SmallGroupDatabase ();
G:=SmallGroup(D,168,42);
G;
FPGroup(G);
G<a,b>:=Group<a,b|a^2,b^3,(a*b)^7,(b*a*b^(-1)*a)^4>;
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
SL := Subgroups(G1);
T := {X`subgroup: X in SL};
#T;
TrivCore := {H:H in T| #Core(G1,H) eq 1};
#TrivCore;
TrivCore;
mdeg := Min({Index(G1,H):H in TrivCore});
Good := {H: H in TrivCore| Index(G1,H) eq mdeg};
#Good;
H := Rep(Good);
#H;
f,G1,K := CosetAction(G1,H);
G1;
s:=IsIsomorphic(G1,N);
s;
G1;

S:=Sym(7);
A:=S!(1, 2)(4, 5);
B:=S!(2, 3, 4)(5, 6, 7);
```

```
N:=sub<S|A,B>;
N eq G1;
N1:=Stabiliser(N,1);
N1;

NN<a,b>:=Group<a,b|a^2,b^3,(a*b)^7,(b*a*b^(-1)*a)^4>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..168]];
for i in [2..168] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

N1:=Stabiliser(N,1);
N1;

for i in [1..168] do if ArrayP[i] eq N!(2, 3, 4)(5, 6, 7)
then print Sch[i];end if; end for;

for i in [1..168] do if ArrayP[i] eq N!(2, 5)(3, 7)
then print Sch[i];end if; end for;

for i in [1..168] do if ArrayP[i] eq N!(3, 6)(4, 5)
then print Sch[i];end if; end for;

for i in [1..168] do if ArrayP[i] eq N!(3, 5)(4, 6)
then print Sch[i];end if; end for;

Centraliser(N,Stabiliser(N,[1,2]));

for i in [1..168] do if ArrayP[i] eq N!(1, 6, 2, 4, 5, 3, 7)
then print Sch[i];end if; end for;

(t*t^a)^i=b^(-1)*a*b*a*b^(-1)*a*b,
(t*t^a)^j=b^(-1)*a*b^(-1)*a*b^(-1)*a*b*a*b*a*b,
```

```
(a*t)^k,
(a*t^(a^b))^l,
(a*t^((a*b^-1)^2))^m,

(a*b^-1*a*t)^n,
(a*b^-1*a*t^a)^o,
(a*b^-1*a*t^(a^(b^2)))^p>;

(a*b^-1*a*b*t)^q,
(a*b^-1*a*b*t^(a^(b^2)))^r,
(a*b^-1*a*b*t^((a*b^-1)^2))^s,

(a*b*t)^u,
((a*b)^3*t)^v

for n,o,p,q,r in [0..5] do
G<a,b,t>:=Group<a,b,t|a^2,b^3,(a*b)^7,(b*a*b^(-1)*a)^4,
t^2,(t,b),(t,a*b*a*b^-1*a),
(a*b^-1*a*t)^n,
(a*b^-1*a*t^a)^o,
(a*b^-1*a*t^(a^(b^2)))^p,
(a*b^-1*a*b*t)^q,
(a*b^-1*a*b*t^(a^(b^2)))^r>;
n,o,p,q,r, #G;
end for;

C:=Classes(N);
C;

C2:=Centraliser(N,N!(1, 2)(4, 5));
C2;
Orbits(C2);
C3:=Centraliser(N,N!(1, 5, 3)(4, 7, 6));
C3;
Orbits(C3);
C4:=Centraliser(N,N!(1, 6, 2, 3)(4, 5));
C4;
Orbits(C4);
C5:=Centraliser(N,N!(1, 3, 4, 6, 7, 5, 2));
C5;
Orbits(C5);
C6:=Centraliser(N,N!(1, 6, 2, 4, 5, 3, 7));
C6;
Orbits(C6);
```

```
for n,o,p,q,r in [0..10] do
G<a,b,t>:=Group<a,b,t|a^2,b^3,(a*b)^7,(b*a*b^(-1)*a)^4,
t^2,(t,b),(t,a*b*a*b^-1*a),
(a*b^-1*a*t)^n,
(a*b^-1*a*t^a)^o,
(a*b^-1*a*t^(a^(b^2)))^p,
(a*b^-1*a*b*t)^q,
(a*b^-1*a*b*t^(a^(b^2)))^r>;
if Index(G,sub<G|a,b>) ge 3 then n,o,p,q,r,
Index(G,sub<G|a,b>);
end if; end for;

nano 2star7N168a
nohup magma "2star7N168a" &> 2star7N168a.out &
nano 2star7N168a.out
```

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